Relativistic hydrodynamics of a superfluid

V. V. Lebedev and I. M. Khalatnikov

L. D. Landau Institute for Theoretical Physics of the USSR Academy of Sciences, Moscow
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We consider a relativistic generalization of the phenomenology describing a superfluid. Thermodynamic identities are derived for the pressure and energy density. A variational principle is constructed from which the nondissipative two-velocity equations of relativistic hydrodynamics can be derived. The equations are also expressed in Hamiltonian form. An expression for the energy-momentum tensor is found. The dissipative terms in the hydrodynamic equations are discussed. The low-velocity and zero-temperature limits are considered in detail. In the latter limit the equations for a rotating relativistic superfluid are derived and the normal modes of the vortex lattice are discussed. It is shown how to generalize the equations to the presence of a gravitational field. An application of the formalism to neutron stars is discussed.

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INTRODUCTION

The recent intensive investigation of pulsars has allowed one to determine the parameters of neutron stars (Refs. 1 and 2). In particular, they have sizes of the order $r \sim 10$ km and an internal liquid core with a density $\rho \sim 10^{14}$–10^{15} g/cm^3. In this case the ratio of the gravitational radius r_g to the radius r of the star is of the order unity, so that neutron stars produce a large gravitational field and general-relativistic effects must be taken into account in their discussion.

It is firmly established that a superfluid phase produced by Cooper pairing of nucleons exists in the core of neutron stars, and that quantized vortices occuring in this phase play an important role in physics of neutron stars. Based on an estimate according to the BCS theory,[3] the transition temperature is $T_c \sim 1$ MeV, and the temperature of the neutron star has the same order of magnitude. In such a situation one must also take into account the motion of the normal component of the fluid and so we must make use of two-velocity hydrodynamics.

The high density of the core of the neutron star has the effect that the Fermi velocity v_F of a nucleon becomes of the order of the speed of light c. Moreover, the speed of sound $c_s \sim v_F$ also becomes of the order of c. The Landau criterion for a superfluid implies that the critical velocity (unrelated to vortex formation) is determined in order of magnitude by the speed of sound c_s i.e., it reaches relativistic magnitude for the superfluid phase in a neutron star, which in turn may play a role in the dynamics of the star. In addition, near the cores of vortices the superfluid speed v_s also becomes of order c. It should be noted that for the description of nonlinear hydrodynamical processes the nonlinear terms coming from the expansion in v/c_s must be taken into account. However, since $c_s \sim c$ terms of the same order are obtained by expanding the equations in the relativistic parameter v/c. All this points to the need for including relativistic effects in the equations of hydrodynamics for the superfluid phase in neutron stars.

The problem of construction of a relativistic hydrodynamics of a two-velocity fluid has already been discussed in the literature. The phenomenology developed by Israel[4] has to be considered inappropriate. First, as is known from the theory of superfluidity,[5] the superfluid component cannot be considered as a completely independent fluid, and second, one must consider as physically inadequate a discussion of a superfluid velocity which is not the gradient of the phase of the Bose condensate. In the present paper we construct, on the basis of a generalization of the model developed by Khalatnikov[6] for He II, the equations of hydrodynamics of a relativistic superfluid, valid also in the presence of a strong gravitational field.

THE SUPERFLUID VELOCITY

Compared to the classical case, a superfluid exhibits an additional hydrodynamical variables: the superfluid viscosity $\mathbf{v}_s$. This variable is related to the phase of the wave function of the Bose-Einstein condensate, so that (in the absence of vortices) we have

$$\mathbf{v}_s = \nabla\alpha. \tag{1}$$

We note that in distinction from the normal velocity, the superfluid velocity (1) which we have defined is not related to the space components of a unit four-vector, and thus can exceed the speed of light c. The quantity which cannot be larger than c is the mass convection velocity defined by the ratio $\mathbf{j}/\rho$ (ρ is the mass density, $\mathbf{j}$ is the mass flux density). This is equivalent to the condition $\rho^2c^2 - \mathbf{j}^2 > 0$, which on account of its Lorentz invariance can be tested in the reference frame where $\mathbf{j} = 0$, where it is trivially true.

The definition (1) allows one to derive an auxiliary relation which will be useful in the sequel. Let $L(\nabla_\mu q, q)$ denote the microscopic Lagrangian density of the system, assumed to be invariant under an internal symmetry group with generator G. According to Noether's theorem this leads to the existence of a conserved quantity with the 4-density

$$j^\mu = -(\partial L/\partial\nabla_\mu q)\,Gq. \tag{2}$$

The energy-momentum transfer

$$T_\nu{}^\mu = (\partial L/\partial\nabla_\mu q)\,\nabla_\nu q - L\delta_\nu{}^\mu \tag{3}$$

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will also be invariant with respect to G. We now perform the local (point-dependent) transformation

$$\delta q=\delta\alpha(x)Gq. \tag{4}$$

Taking into account the fact that the Lagrangian for spinor fields is linear in the gradients of the q, it follows from Eq. (3) that

$$\delta T_\nu^{\ \mu}=j^\mu\nabla_\nu\delta\alpha-\delta_\nu^{\ \mu}j^\lambda\nabla_\lambda\delta\alpha. \tag{5}$$

For the energy density $E=T_0^{\ 0}$ and for the momentum density $g_i^{\ 0}$, the relation (5) yields

$$\delta E=\mathbf{j}\nabla\delta\alpha,\quad \delta\mathbf{g}=\rho\nabla\delta\alpha. \tag{6}$$

In these equations we have introduced the three-dimensional notations $j^\mu=(\rho,\mathbf{j})$.

Now let G be the generator of the (constant) gauge transformations. Then the local gauge transformation (4) leads to a point-dependent change of the phase of the Bose-Einstein condensate, so that, with appropriate normalization, a change is induced in the superfluid velocity:

$$\delta\mathbf{v}_s=\nabla\delta\alpha. \tag{7}$$

The invariance of the Lagrangian with respect to the group of (global) gauge transformations leads to the conservation of the number of particles, and in the adopted normalizations ρ and $\mathbf{j}$ agree apart from the mass factor m with the particle-number density and the particle-flux density. Taking into account Eq. (7) the relation (6) can be rewritten in the form

$$\delta\hat{E}=\hat{\mathbf{j}}\delta\mathbf{v}_s,\quad \delta\hat{\mathbf{g}}=\hat{\rho}\delta\mathbf{v}_s. \tag{8}$$

Here the energy, momentum, mass and mass-flux densities have been replaced by their second-quantized counterparts, which are denoted with a caret above the appropriate letter.

THERMODYNAMIC IDENTITIES

For the derivation of thermodynamic identities we shall start from a local Gibbs (grand canonical) ensemble, determined by the distribution function

$$\exp\left\{\int d^3x\frac{1}{T}(-P-\hat{E}+\mathbf{v}_n\hat{\mathbf{g}}+\mu\hat{\rho})\right\}. \tag{9}$$

This expression contains the following local functions: $P(x)$ is the pressure, $T(x)$ is the local temperature, $\mathbf{v}_n(x)$ is the normal (nonsuperfluid) velocity, and $\mu(x)$ is the local chemical potential. For constant P, T, $\mathbf{v}_n$, and μ the distribution (9) becomes the usual Gibbs distribution,[7] and for a weakly inhomogeneous medium it yields a good approximation of the distribution function. The entropy defined as the negative of the logarithm of the distribution function will have, according to Eq. (9) the following density:

$$s=T^{-1}(P+E-\mathbf{v}_n\mathbf{g}-\mu\rho). \tag{10}$$

Here $E(x)$ is the energy density, $\mathbf{g}(x)$ is the momentum density, and $\rho(x)$ is the mass density, all understood as expectation values with respect to the density matrix (9) of the corresponding second-quantized operators.

The operator part of the integrand of (9) can be written as the zero-component of the following four-vector expression:

$$-\beta^\nu\hat{T}_\nu^{\ \mu}+(\mu/T)\hat{j}^\mu.$$

Here $\hat{T}_\nu^{\ \mu}$ is the second-quantized energy-momentum tensor and $\beta^\mu=(T^{-1},T^{-1}\mathbf{v}_n)$ is the "inverse temperature four-vector."*

The distribution function must be invariant, implying the invariance of μ/T and the four-vector character of β^μ. The part of the argument of the exponential in Eq. (9) involving the pressure can be written in the form

$$-\int dS_\mu\beta^\mu P,$$

where dS^μ denotes the integration element of a timelike hypersurface; this demonstrates the invariant of the pressure P.

The thermdynamic identity for the pressure P can be obtained by varying the normalization condition for the distribution (9) with respect to the local T, $\mathbf{v}_n$ and μ as well as with respect to the superfluid velocity $\mathbf{v}_s$, according to Eq. (8). Taking into account the equation (10) we obtain

$$dP=\rho d\mu+sdT+\mathbf{g}d\mathbf{v}_n-(\mathbf{j}-\rho\mathbf{v}_n)d\mathbf{v}_s. \tag{11}$$

This expression involves the mass flux density $\mathbf{j}$ which in the relativistic case is not equal to the momentum density $\mathbf{g}$, and this is the only difference between (11) and the nonrelativistic case. One can rewrite the identity (11) in a covariant form. For this purpose we introduce the quantity $\mathbf{w}$ via the definition

$$\mathbf{g}=\rho\mathbf{v}_s+s\mathbf{w}. \tag{12}$$

We now construct the quantities

$$v_\mu=(\mu+\mathbf{v}_s\mathbf{v}_n,-\mathbf{v}_s),\quad w_\mu=(T+\mathbf{v}_n\mathbf{w},-\mathbf{w}). \tag{13}$$

In terms of these quantities Eq. (11) can be rewritten in the form

$$dP=j^\mu dv_\mu+s^\mu dw_\mu. \tag{14}$$

Here $j^\mu=(\rho,\mathbf{j})$ is the density of the mass 4-flux and $s^\mu=(s,s\mathbf{v}_n)$ is the density of the entropy 4-current. In view of the four-vector character of these quantities and the invariance of the pressure P, the quantities introduced in (13) are also four-vectors. On account of the collinearity of the four-vectors s^μ and β^μ and of the identity $\beta^\mu w_\mu=1$, the identity (14) can be reduced to the form

$$dP=j^\mu dv_\mu-Tsw_\mu d\beta^\mu. \tag{15}$$

Here $Ts=w_\mu s^\mu$ is an invariant, so that covariance of the identity (15) is ensured.

Extracting the energy density E from the relation (10) we obtain, in terms of $v_0=\mu+\mathbf{v}_s\cdot\mathbf{v}_n$ and $w_0=T+\mathbf{w}\cdot\mathbf{v}_n$ and taking Eq. (12) into account,

$$E=\rho v_0+sw_0-P. \tag{16}$$

The identity (16) is a Legendre transformation from the variables v_0 and w_0 to the variables ρ and s. Thus the energy density is a function $E(\rho, s, \mathbf{v}_s, w)$ whose differential can be obtained from Eqs. (14) and (16) in the following form:

$$dE = v_0 d\rho + w_0 ds - \mathbf{j} d\mathbf{v}_s - s\mathbf{v}_n d\mathbf{w}. \qquad (17)$$

We note that the identities (14) and (15) remain valid also in the presence of a gravitational field[8]; in this case the quantities v_μ and w_μ have to be considered as covectors (coefficients of 1-forms) whereas the other four-vectors are contravariant.

A VARIATIONAL PRINCIPLE

The nondissipative hydrodynamic equations of a relativistic superfluid can be derived from a least-action principle for the macroscopic degrees of freedom of the system. As can be seen from the structures of the energy-momentum tensor (2), the pressure P must coincide with the Lagrangian density L on account of the equations of motion. To obtain L from the function $P(v_\mu, w_\mu)$ we must substitute in the latter

$$v_\mu = -\nabla_\mu \alpha, \qquad (18)$$

$$w_\mu = -\nabla_\mu \xi - \varphi \nabla_\mu \gamma. \qquad (19)$$

The meaning of the variables $\alpha, \xi, \varphi, \gamma$ in L can be understood from the spatial components of Eqs. (18) and (19). For v_μ these components coincide with Eqs. (1), i.e., α is the phase of the wave function describing the Bose-Einstein condensate. For the covector w_μ the spatial component is

$$\mathbf{w} = \nabla \xi + \varphi \nabla \gamma. \qquad (20)$$

Thus, the variables ξ, φ, and γ correspond to the three independent degrees of freedom of w. This situation is similar to the case of He II,[6] where for the description of the three degrees of freedom of the normal momentum density it was also necessary to introduce Clebsch variables of the type of φ and γ.

Variation with respect to α, ξ, β, and γ yields the following equations

$$\nabla_\mu j^\mu = 0, \qquad (21)$$

$$\nabla_\mu s^\mu = 0, \qquad (22)$$

$$\nabla_\mu (\varphi s^\mu) = 0, \quad s^\mu \nabla_\mu \gamma = 0. \qquad (23)$$

The equations (21) and (22) are respectively the conservation laws of mass (i.e., particle number), and entropy. On account of Eq. (22) the equation for φ can be written in the form

$$s^\mu \nabla_\mu \varphi = 0. \qquad (24)$$

Thus, the equations for the Clebsch variables φ and γ are of the same type and represent transport equations with the normal velocity. Taking into account Eqs. (23) and (24), the following equations can be derived from the representations (18) and (19):

$$\nabla_\mu v_\nu - \nabla_\nu v_\mu = 0, \qquad (25)$$

$$s^\mu (\nabla_\mu w_\nu - \nabla_\nu w_\mu) = 0. \qquad (26)$$

Combined with the equation for the superfluid velocity $\mathbf{v}_s$, the relation (25) contains the condition $\operatorname{curl} \mathbf{v}_s = 0$; the relation (26) is in fact an equation for the quantity $\mathbf{w}$, since only three of its four components are independent [by multiplying (26) with s^ν one obtains an identity]. Equation (25) for the superfluid four-velocity has in fact been considered by Rothen[9] and by Israel,[4] who in place of $\mathbf{v}_\mu$ have used the notation $\mu_0 u_\mu$, where u_μ is a normalized four-vector, and μ_0 is the invariant chemical potential.

The relations (18), (19), and (21)–(26) remain valid also in the presence of a gravitational field. In this case the derivatives ∇_μ should be interpreted as covariant derivatives. Christoffel symbols make their appearances only in Eqs. (21) and (22) and are absent from the other equations since α, ξ, φ, and γ are scalars and the other quantities are differential forms (i.e., antisymmetric covariant tensors).

THE HAMILTONIAN FORMALISM

One can also reformulate the nondissipative equations we have found for a relativistic superfluid in Hamiltonian language. We write out this formulation directly in the presence of a gravitational field, where, in accordance with what was said above, the hydrodynamic action functional has the form

$$\int d^4x (-g)^{1/2} L(\nabla_\mu \alpha, \nabla_\mu \xi, \nabla_\mu \gamma, \varphi). \qquad (27)$$

Here $g = c^{-2} \det g_{\mu\nu}$, since the Lorentz metric in the normalization we have adopted has the form

$$g^{(0)}_{\mu\nu} = \begin{pmatrix} c^2 & 0 \\ 0 & -\delta_{ik} \end{pmatrix}.$$

Thus, for the system under consideration the generalized coordinates are α, ξ, and γ, and the corresponding canonical momenta, taking into account Eqs. (14), (18), (19), and (27), are

$$p_\alpha = (\partial/\partial\dot{\alpha}) \sqrt{-g} L = -\sqrt{-g}\rho,$$

$$p_\xi = (\partial/\partial\dot{\xi}) \sqrt{-g} L = -\sqrt{-g} s,$$

$$p_\gamma = (\partial/\partial\dot{\gamma}) \sqrt{-g} L = -\sqrt{-g} s\varphi. \qquad (28)$$

As usual, the Hamiltonian density is

$$H = p_\alpha \dot{\alpha} + p_\xi \dot{\xi} + p_\gamma \dot{\gamma} - \sqrt{-g} L. \qquad (29)$$

However, H must be expressed in terms of the canonical variables p and q. This is realized if one takes into account the zero-components of the substitutions (18) and (19). As a result we obtain the Hamiltonian function

$$\mathscr{H} = \int d^3x \sqrt{-g}\, E(\rho, s, v_s, w). \qquad (30)$$

Here ρ and must be expressed in terms of the p_α and p_ξ of Eq. (28), $\mathbf{v}_s$ is expressed in terms of Eq. (1), and for $\mathbf{w}$ one must use the expression (20), substituting into it $\varphi = p_\gamma / p_\xi$ in accord with Eq. (28).

The Hamiltonian equations for pairs of canonically conjugate variables (p_α, α), (p_ξ, ξ), and (p_γ, γ) account being taken of Eq. (14), have the form:

$$(\partial/\partial t)\gamma = \delta\mathscr{H}/\delta p_\gamma = -\mathbf{v}_n \nabla \gamma, \qquad (31)$$

$$(\partial/\partial t)\alpha = \delta\mathscr{H}/\delta p_\alpha = -v_0 = -\mu - \mathbf{v}_n \nabla \alpha, \qquad (32)$$

$$(\partial/\partial t)\,\xi=\delta\mathscr{H}/\delta p_\xi=-w_0+\varphi v_n\nabla\gamma=-T-v_n\nabla\xi, \tag{33}$$

$$(\partial/\partial t)\,(\sqrt{-g}s\varphi)=\delta\mathscr{H}/\delta\gamma=-\nabla\,(\sqrt{-g}s\varphi v_n), \tag{34}$$

$$(\partial/\partial t)\,(\sqrt{-g}\rho)=\delta\mathscr{H}/\delta\alpha=-\nabla\,(\sqrt{-g}\mathbf{j}), \tag{35}$$

$$(\partial/\partial t)\,(\sqrt{-g}s)=\delta\mathscr{H}/\delta\xi=-\nabla\,(\sqrt{-g}s v_n). \tag{36}$$

In transforming the right-hand sides of Eqs. (32) and (33), use has been made of the expressions (13) and (20). The equations (31) and (34) are equivalent to the equations (23), the equations (31) and (32) are equivalent to the zero-components of the substitutions (18) and (19), and the equations (35) and (36) are equivalent to Eqs. (21) and (22) (with covariant derivatives). Taking account of Eq. (36), the equation (34) yields for φ an equation equivalent to (24):

$$\partial\varphi/\partial t=-v_n\nabla\varphi. \tag{37}$$

These Hamiltonian equations allow us to write down the equations of motion for $\mathbf{v}_s$ and $\mathbf{w}$. Taking into account Eq.(1) we obtain from (32) for $\mathbf{v}_s$ the equation

$$\partial\mathbf{v}_s/\partial t=-\nabla v_0=-\nabla\,(\mu+\mathbf{v}_s\mathbf{v}_n). \tag{38}$$

This equation is equivalent to the zero component of (25). Taking (20) into account we obtain from Eqs. (31), (33), (37), for $\mathbf{w}$

$$\partial\mathbf{w}/\partial t=-\nabla w_0+[\mathbf{v}_n\times[\nabla\times\mathbf{w}]]=-\nabla\,(T+\mathbf{v}_n\mathbf{w})+[\mathbf{v}_n\times[\nabla\times\mathbf{w}]]. \tag{39}$$

This equation is equivalent to the spatial part of the Eq. (26). The system (35), (36), (38), and (39) is a complete set of equations for the relativistic superfluid, since the macroscopic state of such a fluid at a given time is uniquely characterized by prescribing the spatial distributions of ρ, s, $\mathbf{v}_s$, and $\mathbf{w}$. The right-hand sides of these equations can be expressed in terms of the derivatives of the functions $E\,(\rho, s, \mathbf{v}_s, \mathbf{w})$ according to the identity (17). Thus, prescribing this function is sufficient for the expression of the time-derivatives of the quantities ρ, s, $\mathbf{v}_s$, and $\mathbf{w}$ in terms of these same quantities.

We note that the Hamiltonian formalism described here is analogous to the Hamiltonian formalism developed by Polrovskiĭ and Khalatnikov[10] for the description of superfluid ^{4}He.

THE ENERGY-MOMENTUM TENSOR

The equations we have found lead to the energy and momentum conservation equations

$$\nabla_\mu T_\nu{}^\mu=0. \tag{40}$$

The energy-momentum tensor $T_\nu{}^\mu$ for the hydrodynamical degrees of freedom can be constructed according to Eq. (3). Taking acocunt of the identity (14) and of the relations (18) and (19), we obtain

$$T_\nu{}^\mu=j^\mu v_\nu+s^\mu w_\nu-P\delta_\nu{}^\mu. \tag{41}$$

Equation (40) now follows from Eqs. (21), (22), (25), and (26) if the identity (14) is taken into account. From Eq. (41) we obtain for the energy density $E=T_0{}^0$ an expression identical to (16), and for the momentum density $g_i - T_i{}^0$ an expression identical to (12). The components $Q^i=T_0{}^i$ and $\Pi_i{}^k=-T_i{}^k$ of the tensor (41) represent the energy of flux density (Poynting vector) and the stress tensor.

Multiplying (41) by β^ν and recognizing that $\beta^\nu w_\nu=1$, we obtain the expression for the entropy 4-flux density

$$s^\mu=P\beta^\mu+\beta^\nu T_\nu{}^\mu-j^\mu\beta^\nu v_\nu. \tag{42}$$

Note that expression (10) is the zero-component of Eq. (42). Differentiating (42), account being taken of the identity (15), of the expression (41), and of collinearity of β^μ and s^μ, we find

$$ds^\mu=-(\mu/T)\,dj^\mu+\beta^\nu dT_\nu{}^\mu+(\beta^\mu j^\nu-\beta^\nu j^\mu)\,dv_\nu. \tag{43}$$

The relations (42) and (43) are direct generalizations to the case of a superfluid of expressions for a classical fluid[11] [in the case of a classical fluid the last term is absent in Eq. (43)].

On account of the invariant nature of P it must be a function of invariants:

$$P=\Psi\,(I_1,\ I_2,\ I_3). \tag{44}$$

Here the invariants I_i have the following form:

$$I_1={}^1/{}_2 g^{\mu\nu}v_\mu v_\nu,\qquad I_2=g^{\mu\nu}v_\mu w_\nu,\qquad I_3={}^1/{}_2 g^{\mu\nu}w_\mu w_\nu. \tag{45}$$

Calculating the derivatives of (14) according to Eq. (14), we obtain:

$$j^\mu=(\partial\Psi/\partial I_1)\,v^\mu+(\partial\Psi/\partial I_2)\,w^\mu, \tag{46}$$

$$s^\mu=(\partial\Psi/\partial I_2)\,v^\mu+(\partial\Psi/\partial I_3)\,w^\mu. \tag{47}$$

The spatial components of Eqs. (46) and (47) yield expressions for $\mathbf{j}$ and $s\mathbf{v}_n$ in terms of $\mathbf{v}_s$ *and* $\mathbf{w}$. Taking into account also the relation (12) for $\mathbf{g}$ we find the expressions for all the vector quantities of the problem in terms of the two independent vectors $\mathbf{v}_s$ and $\mathbf{w}$.

Substituting (47) into (41) and omitting the superscript μ, we obtain

$$T_{\mu\nu}=\frac{\partial\Psi}{\partial I_1}v_\mu v_\nu+\frac{\partial\Psi}{\partial I_2}(v_\mu w_\nu+v_\nu w_\mu)+\frac{\partial\Psi}{\partial I_3}w_\mu w_\nu-\Psi g_{\mu\nu}. \tag{48}$$

Thus, the energy-momentum tensor constructed here is symmetric. We note that if one takes into account $L=P$ and the structure of the expressions (18) and (19), it becomes clear that the energy-momentum tensor (48) constructed by means of the canonical procedure (3) coincides with the "gravitational" energy-momentum tensor[8]

$$T_{\mu\nu}=2\partial L/\partial g^{\mu\nu}-Lg_{\mu\nu}.$$

All equations in this section are valid in the presence of a gravitational field, with ∇_μ in Eq. (40) to be interpreted as a covariant derivative. To close the system of equations in the presence of a gravitational field one must add to the equations considered the Einstein equation with the energy-momentum tensor (48).

THE DISSIPATIVE TERMS

We shall consider a weakly non-equilibrium state of a superfluid close to the state determined by a nonhomogeneous Gibbs (grand canonical) distribution, so that we shall again characterize the hydrodynamic state of the system by the four-vector β^μ describing the (inverse) local temperature and normal velocity, as well as the quantities v_μ related to the gradient of the phase α by the relations (18). The hydrodynamic equations including dissipation have again the form of conservation laws of energy-momentum (40) and mass (21), which yields five equations for the quantities β^μ, α (a second-order equation for the latter). However, the energy-momentum tensor and the mass 4-flux vector density contain in addition to the nondissipative terms (labeled by the subscript r) the dissipative additions (denoted by the subscript d):

$$T_\nu^{\ \mu}=T_{r\nu}^{\ \ \mu}+T_{d\nu}^{\ \ \mu}, \qquad j^\mu=j_r^{\ \mu}+j_d^{\ \mu}. \tag{49}$$

The nondissipative quantities are expressed in terms of β^μ and $\nabla_\mu\alpha$ according to Eq. (41), the identity (15), and the expression (18). The dissipative terms are expressed in terms of the derivatives of β^ν and $\nabla_\mu\alpha$.

We note that one may impose four arbitrary conditions on the dissipative additions $T_{d\nu}^{\ \ \mu}$ and $j_D^{\ \mu}$, due to the possibility of redefining the local inverse four-temperature β^μ. This is where the situation differs from the case of a classical fluid,[11] where there are five such conditions. The difference is due to the fact that the chemical potential μ is fixed by Eq. (32) and cannot be redefined.

Let us calculate the quantity $\nabla_\mu s_r^{\ \mu}$ according to the identity (43) with (18) taken into account. Considering that the conservation laws are valid for the total quantities (49), we obtain

$$\nabla_\mu[s_r^{\ \mu}-(\mu/T)j_d^{\ \mu}+\beta^\nu T_{d\nu}^{\ \ \mu}]=-j_d^{\ \mu}\nabla_\mu(\mu/T)+T_{d\nu}^{\ \ \mu}\nabla_\mu\beta^\nu. \tag{50}$$

The second and third terms in the left-hand side of Eq. (50) yield the density of the entropy 4-flux, and the right-hand side of Eq. (50) is an expression for the entropy production, i.e., must be positive-definite.

In the linear approximation we have

$$\begin{pmatrix} j_d^{\ \mu} \\ T_{d\nu}^{\ \ \mu}\end{pmatrix}=\begin{pmatrix} \zeta^{\mu\lambda} & \zeta_\varkappa^{\ \mu\lambda} \\ \zeta_\nu^{\ \mu\lambda} & \zeta_{\nu\varkappa}^{\ \mu\lambda}\end{pmatrix}\begin{pmatrix} -\nabla_\lambda(\mu/T) \\ \nabla_\lambda\beta^\varkappa\end{pmatrix} \tag{51}$$

Here ζ is the matrix of kinetic coefficients. On account of the Onsager reciprocity relations the matrix ζ is symmetric. i.e.,

$$\zeta^{\mu\lambda}=\zeta^{\lambda\mu}, \qquad \zeta_\varkappa^{\ \mu\lambda}=\zeta_{\ \varkappa}^{\lambda\mu}, \qquad \zeta_{\lambda\varkappa}^{\ \mu\nu}=\zeta_{\varkappa\lambda}^{\ \nu\mu},$$

and the following relations must hold on account of the symmetry of the energy-momentum tensor:

$$\zeta_\nu^{\ \mu\lambda}=g^{\mu\xi}\zeta_\xi^{\ \varkappa\lambda}g_{\nu\varkappa}, \qquad \zeta_{\nu\rho}^{\ \mu\lambda}=g^{\mu\xi}\zeta_{\xi\rho}^{\ \varkappa\lambda}g_{\varkappa\nu}. \tag{52}$$

In addition, we must require positive-definiteness of the quadratic form defined by the matrix ζ. The number of invariants of the matrix ζ can be counted if one takes into account its symmetry and the fact that there exist two distinguished directions related to the four-vectors β^μ and $\nabla_\mu\alpha$. As a result we obtain 28 invariants (4 for $\zeta^{\mu\nu}$, 10 for $\zeta_\varkappa^{\ \mu\lambda}$ and 14 for $\zeta_{\lambda\varkappa}^{\ \mu\nu}$).

We note that all equations obtained in the present section remain in force in the presence of a gravitational field if the derivatives ∇_μ and in Eqs. (50) and (51) are interpreted as covariant derivatives.

NONRELATIVISTIC VELOCITIES

In the present section we consider the motion of the fluid with nonrelativistic velocities $\mathbf{v}_s$ and $\mathbf{v}_n$. We note that even in this limit the equations of motion of the superfluid phase of a neutron star do not go over into the equations for an ordinary superfluid,[5] since on account of the high density the microscopic velocities of the motion of nucleons relativistic. This means, in particular, that the equality $\mu\simeq c^2$ does not hold, and only the order-of-magnitude estimate $\mu\simeq c^2$ remains valid.

We consider the pressure as a function of the invariants

$$P=\Phi(\mu_0, T_0, I_0), \tag{53}$$

where

$$\mu_0=c(g^{\mu\nu}v_\mu v_\nu)^{1/2}, \qquad T_0=c(g_{\mu\nu}\beta^\mu\beta^\nu)^{-1/2}, \qquad I_0=(\mu/T)T_0-\mu_0. \tag{54}$$

Here μ_0 is the invariant chemical potential, T_0 is the invariant temperature, and I_0 plays the role of the invariant square of the relative velocity. If the conditions $v_s,\ v_n\ll c$ hold, and in the absence of a gravitational field,[1)] we have

$$\mu_0\approx\mu+v_nv_s-\frac{c^2}{\mu}\frac{v_s^2}{2}, \qquad T_0\approx T, \qquad I_0\approx\frac{\mu}{2c^2}\left(\mathbf{v}_n-\frac{c^2}{\mu}\mathbf{v}_s\right)^2. \tag{55}$$

In view of $\mu\sim c^2$ we find, in accord with Eq. (11), and taking into account Eqs. (53) and (55) to first order in $(v/c)^2$:

$$\rho=\partial\Phi/\partial\mu_0, \qquad s=\partial\Phi/\partial T_0, \tag{56}$$

$$\mathbf{g}=\rho\mathbf{v}_s+(\partial\Phi/\partial I_0)[(\mu/c^2\mathbf{v}_n-\mathbf{v}_s)]\equiv\rho\mathbf{v}_s+\mathbf{g}_n, \tag{57}$$

$$\mathbf{j}=(c^2/\mu)\mathbf{g}. \tag{58}$$

We now note that the mass flux density $\mathbf{j}$ and the momentum flux density $\mathbf{g}$ do not coincide even at nonrelativistic velocities v_s and v_n.

We now discuss the energy density E given by Eq. (16). Making use of Eqs. (56)–(58), we obtain

$$E=(c^2/\mu)(\rho v_s^2/2+\mathbf{v}_s\mathbf{g}_n)+\varepsilon. \tag{59}$$

Here g_n is the normal-momentum density introduced in Eq. (57), and

$$\varepsilon=\mu_0\rho+Ts+[\mathbf{v}_n-(c^2/\mu)\mathbf{v}_s]\mathbf{g}_n-P. \tag{60}$$

In agreement with Eqs. (60) and (55)–(57) the differential of ε is

$$d\varepsilon=\mu_0d\rho+Tds+[\mathbf{v}_n-(c^2/\mu)\mathbf{v}_s]d\mathbf{g}_n. \tag{61}$$

We note that the expression (59) is in agreement with the fact that under Galilei transformations with generator $\mathbf{v}$:

$$\delta\mathbf{v}_s=(\mu/c^2)\delta\mathbf{v}, \qquad \delta\mathbf{v}_n=\delta\mathbf{v}. \tag{62}$$

The relations (62) follow from the Lorentz transformations for the four-vectors v_μ and β^μ for small v_s and v_n. The expressions (58)–(61) generalize the relations for the nonrelativistic superfluid and go over into the latter in the limit $\mu\approx c^2$.

If the motion of the superfluid takes place on a uniform background we can separate the chemical potential into two contributions: $\mu = \bar{\mu} + \mu'$, where the constant $\bar{\mu}$ is of the order of c^2 and $\mu' \sim v^2$ ($\bar{\mu} = c^2$ for He II). In this case one may exclude from consideration the rest energy by means of the following transformation of the energy density E and the energy flux density $\mathbf{Q}$:

$$E' = E - \bar{\mu}\rho, \quad \mathbf{Q}' = \mathbf{Q} - \bar{\mu}\mathbf{j}. \tag{63}$$

As a result, the large quantities of the order of c^2 disappear from the equations and the ratio $c^2\mu^{-1} \approx c^2\bar{\mu}^{-1}$ becomes a constant of the order of units. We note that as a result of the transformation (63) the condition $\mathbf{Q} = \mathbf{g}c^2$, which follows from the symmetry of the energy-momentum tensor, will be violated.

As regards the dissipative terms, in the case of small v_s and v_n there remains in 4-space only one distinguished direction $u^\mu = (1,0)$. Accordingly the number of independent invariants of the matrix ζ reduces to ten, to wit: two for $\zeta^{\mu\nu}$, three for $\zeta^{\mu\nu}_{\lambda}$, and five for $\zeta^{\mu\nu}_{\lambda\varkappa}$. Since we can redefine β^μ, we require that the dissipative additions to the energy and momentum densities should vanish, i.e., $u_\mu T^{\mu}_{d\nu} = 0$. This condition removes five more constants, and as a result we find

$$\begin{gathered}\zeta^{\mu\nu} = \zeta_1 u^\mu u^\lambda + \zeta_2 g_\perp^{\mu\lambda}, \qquad \zeta_\nu^{\mu\lambda} = \zeta_3 \delta_{\perp\nu}^{\mu} u^\lambda, \\ \zeta_{\nu\varkappa}^{\mu\lambda} = \zeta_4 \delta_{\perp\nu}^{\mu} \delta_{\perp\varkappa}^{\lambda} + \zeta_5 (g_\perp^{\mu\lambda} g_{\perp\nu\varkappa} + \delta_{\perp\varkappa}^{\mu} \delta_{\perp\nu}^{\lambda}).\end{gathered} \tag{64}$$

Here we have introduced the notation $g_\perp^{\mu\nu} = g^{\mu\nu} - c^{-2}u^\mu u^\nu$ and similar a notation for the Kronecker deltas. Substituting Eq. (64) into (51) we obtain

$$\begin{gathered}\rho_d = -\zeta_1 \frac{\partial}{\partial t}\frac{\mu}{T} + \zeta_3 \nabla \frac{\mathbf{v}_n}{T}, \qquad \mathbf{j}_d = c^{-2}\zeta_2 \nabla \frac{\mu}{T}, \\ \Pi_{dik} = \zeta_3 \delta_{ik} \frac{\partial}{\partial t}\frac{\mu}{T} - \zeta_4 \delta_{ik} \nabla \frac{\mathbf{v}_n}{T} - \zeta_5 \left(\nabla_i \frac{v_{nk}}{T} + \nabla_k \frac{v_{ni}}{T}\right).\end{gathered} \tag{65}$$

We note that the dissipative addition to $\mathbf{j}$ is small in the parameter v^2/c^2 and in the approximation adopted here we should have neglected it (so that the equality should remain valid even when dissipation is taken into account). However, we have retained $\mathbf{j}_d$ since under the transformation (63) Q' acquires on account of $\mathbf{j}_d$ a dissipative addition which is not small.

In distinction from the formalism considered here, in the traditional model the mass density ρ is considered as the independent variable, rather than $\dot{\alpha}$, which is used here. This means that for a transition to the traditional treatment it is necessary to redefine *ab initio* the chemical potential in such a way that $\rho_d = 0$, leading to the appearance of dissipative additions in the right-hand side of Eq. (32). As a result there appear equations with a heat-transport coefficient and four coefficients of first and second viscosity,[5] which is equivalent to the description in terms of the five coefficients of ζ in Eq. (65).

ZERO TEMPERATURE

At zero temperature the entropy vanishes, since the second term in the right-hand side of Eq. (15) disappears. Thus the pressure P will now depend only on v_μ, or more precisely, on the invariant chemical potential μ_0, introduced according to Eq. (54)

$$P = \Phi(\mu_0). \tag{66}$$

Calculating j^μ according to Eq. (15) we obtain

$$j^\mu = (\partial\Phi/\partial\mu_0)(c^2/\mu_0) v^\mu. \tag{67}$$

The only (second-order) equation for the quantity α which can be obtained from a variation of the action (27) has the form of the mass conservation law (21) with the mass 4-flux density (67) (into which one should substitute $v^\mu = -g^{\mu\nu}\nabla_\nu \alpha$).

In the reference frame where $\mathbf{v}_s = 0$ (in the absence of a gravitational field) we have, according to (67) and the definitions (18) and (54),

$$\rho = \partial\Phi/\partial\mu_0.$$

Taking Eq. (66) into account, we find that for the equation of state $P \propto \rho^\gamma$ (which is obtained for various models of the state of matter in neutron stars) the function Φ has the following behavior:

$$\Phi \propto \mu_0^{\gamma/(\gamma-1)}. \tag{68}$$

We note that the relation (68) is no longer dependent on the reference frame and is valid even in the presence of a gravitational field.

In connection with the rapid rotation of neutron stars an important role will be played by quantized vortices in the superfluid phase. In the core of the vortex the potential character (25) of the superfluid four-velocity v_μ will be violated. The circulation of this vector in one cycle around a vortex line will be

$$\oint dx^\mu v_\mu = 2\pi\hbar/m, \tag{69}$$

where m is the mass of the paired particles. If one averages over distances much larger than the spacing between the vortices, the v_μ become independent variables. The vorticity $\nabla_\mu v_\nu - \nabla_\nu v_\mu$ is directly related to the vortex density, and it follows from Eq. (69) that the number of vortices which penetrate through a given surface will be determined by the integral over this surface:

$$N = \frac{m}{2\pi\hbar}\int df^{\mu\nu}(\nabla_\mu v_\nu - \nabla_\nu v_\mu). \tag{70}$$

The presence of continuously distributed vortices thus leads to a dependence of the pressure P on the vorticity $\nabla_\mu v_\nu - \nabla_\nu v_\mu$.

The four equations for the v_μ will now be the conservation laws of energy-momentum (40), and in the presence of vortices the energy-momentum tensor has the form:

$$T_\nu^{\mu} = \frac{\partial P}{\partial v_\mu} v_\nu + 2\frac{\partial P}{\partial(\nabla_\mu v_\lambda - \nabla_\lambda v_\mu)}(\nabla_\nu v_\lambda - \nabla_\lambda v_\nu) - \delta_\nu^{\mu} P. \tag{71}$$

The expression (71) is a generalization of Eq. (41) [at $T = 0$ the second term in Eq. (41) vanishes]. Now Eq. (40) is equivalent to the mass conservation law (21) together with the equation for the superfluid velocity

$$j^\mu(\nabla_\mu v_\nu - \nabla_\nu v_\mu) = 0. \quad (72)$$

[Equation (72) contains only three independent components, since upon multiplication by j^ν it becomes an identity.] The mass 4-flux density j^μ which occurs in Eqs. (21) and (72) equals

$$j^\mu = \partial P/\partial v_\mu + 2\nabla_\lambda \partial P/\partial(\nabla_\mu v_\lambda - \nabla_\lambda v_\mu). \quad (73)$$

The equation (72) is a generalization of the equation for the superfluid velocity used by Rothen[12] for the description of the geometry of vortex lines in the gravitational field of a neutron star.[2]

As a matter of fact, the pressure P will depend on the four-vector ω^μ which is the generalization of curl$\mathbf{v}_s$ to the relativistic case (a factor $(-g)^{1/2}$ appears on account of the gravitational field)

$$\omega^\mu = (c^2\sqrt{-g})^{-1}\varepsilon^{\mu\nu\lambda\rho} v_\nu \nabla_\lambda v_\rho. \quad (74)$$

We note the following identity, which follows from this definition:

$$v_\mu \omega^\mu = 0.$$

Thus P will be a function of only two invariants

$$P = \Phi(\mu_0, \omega). \quad (75)$$

Here μ_0 is given by the expression (54) and

$$\omega = (-g_{\mu\nu}\omega^\mu\omega^\nu)^{1/2}.$$

Substituting the function (75) into the equations (71), (73), we obtain the following expressions:

$$j^\mu = \frac{c^2}{\mu_0}\frac{\partial\Phi}{\partial\mu_0} v^\mu - \frac{1}{c^2\sqrt{-g}}\varepsilon^{\mu\nu\lambda\rho} v_\nu \nabla_\lambda \left(\frac{\partial\Phi}{\partial\omega}\frac{\omega_\rho}{\omega}\right), \quad (76)$$

$$T_\nu{}^\mu = \frac{c^2}{\mu_0}\frac{\partial\Phi}{\partial\mu_0} v^\mu v_\nu + \frac{\omega^\mu\omega_\nu}{\omega}\frac{\partial\Phi}{\partial\omega} - \left(\Phi - \omega\frac{\partial\Phi}{\partial\omega}\right)\delta_\nu{}^\mu. \quad (77)$$

Thus, the presence of vortices leads to the appearance of terms containing ω in the mass 4-flux density and in the energy-momentum tensor. We note that (in the covariant representation) the energy-momentum tensor $T_{\mu\nu}$ given in Eq. (77) is manifestly symmetric.

We now consider the derived equations in a locally-flat coordinate system $v_s \ll c$. In this case, to first order in v_s/c, we obtain from Eq. (74)

$$\rho = \frac{\partial\Phi}{\partial\mu_0}, \quad \mathbf{j} = \frac{c^2}{\mu_0}\frac{\partial\Phi}{\partial\mu_0}\mathbf{v}_s - \frac{\mu_0}{c^2}\,\mathrm{rot}\left(\frac{\partial\Phi}{\partial\omega}\frac{\boldsymbol{\omega}}{\omega}\right). \quad (78)$$

In the same limit

$$v_0 = \mu_0 + (c^2/2\mu_0) v_s^2, \quad \omega = (\mu_0/c^2)\,\mathrm{rot}\,\mathbf{v}_s. \quad (79)$$

Thus, taking into account Eq. (62) we see that the connection of ω with the local angular velocity ω_0 is given by the expression

$$\omega = 2(\mu_0^2/c^4)\omega_0.$$

Equation (72) for the superfluid velocity takes the form

$$\partial\mathbf{v}_s/\partial t = -\nabla v_0 + \rho^{-1}[\mathbf{j}\times\mathrm{rot}\,\mathbf{v}_s]. \quad (80)$$

We now consider small oscillations around the state corresponding to uniform rotation of the fluid with angular velocity ω_0. Linearizing Eq. (80) with allowance for (78) and keeping in mind the condition div$\mathbf{v}_s = 0$, we find for the frequency Ω of the oscillations of the vortex lattice an expression which generalizes the expression for rotating He II (Ref. 13):

$$\Omega = 2\omega_0 - (\mu_0/\rho c^2)(\partial\Phi/\partial\omega)k_z^2. \quad (81)$$

Here k_z is the wave vector along the direction of ω_0. The coefficient of k_z^2 can be obtained from energy considerations. The energy per unit length of a vortex, account being taken of relation (59) and the condition (69), is

$$(\pi c^2\rho\hbar^2/4\mu_0 m^2)\ln(R/a).$$

Here R is the distance between the vortices, and a is the radius of the core of the vortex. In agreement with (70) the number of vortices per unit area equals

$$(m/\hbar)|\mathrm{rot}\,\mathbf{v}_s| = (mc^2/\hbar\mu_0)\omega \sim 1/R^2.$$

Thus, the contribution of vortices to the energy density equals

$$(\pi\hbar\rho c^4/4\mu_0^2 m)\omega\ln(R/a).$$

Consequently, the coefficient in front of k_z^2 in Eq. (81) equals

$$-\frac{\mu_0}{\rho c^2}\frac{\partial\Phi}{\partial\omega} = \frac{\pi\hbar}{4m}\frac{c^2}{\mu_0}\ln\frac{R}{a}.$$

For neutrons,

$$\pi\hbar/4m \approx 0.5\cdot 10^{-3}\ \mathrm{cm^2/sec}.$$

APPENDIX

In the present Appendix we shall show how to construct a variational principle for the relativistic equations of a classical fluid. As is well known, in a classical fluid the mass density ρ and the entropy density are transported with the same speed v (which, in particular, guarantees the existence of an isentropic solution $s/\rho = \mathrm{const}$). Thus the system is characterized only by two invariants

$$s_0 = s(1-v^2/c^2)^{1/2}, \quad \rho_0 = \rho(1-v^2/c^2)^{1/2}. \quad (A1)$$

Accordingly, the pressure P is a function of the invariants

$$P = P(\mu_0, T_0) \quad (A2)$$

satisfying the following identity

$$dP = \rho_0 d\mu_0 + s_0 dT_0. \quad (A3)$$

One must choose as a Lagrangian density the function

$$L = P(u^\mu\nabla_\mu\alpha, u^\mu\nabla_\mu\xi) + \lambda(u^\mu u_\mu - c^2). \quad (A4)$$

Here the variables α and ξ play the same role as in the superfluid, u^μ is to be interpreted as the four-velocity, the Lagrange multiplier λ ensures the normalization condition

$$u^\mu u_\mu = c^2. \quad (A5)$$

Variation with respect to α, ξ, and u^μ together with the identity (A3) leads to the equations

$$\nabla_\mu(\rho_0 u^\mu)=0, \quad \nabla_\mu(s_0 u^\mu)=0, \tag{A6}$$

$$\rho_0\nabla_\mu\alpha+s_0\nabla_\mu\xi+\lambda u_\mu=0. \tag{A7}$$

The equations (A6) represent the mass and entropy conservation laws and (A7) yields an expression for the four-velocity. The multiplier λ is easiest to determine by multiplying Eq. (A7) by u^μ as a result of which, taking into account (A5) and the conditions

$$u^\mu\nabla_\mu\alpha=\mu_0, \quad u^\mu\nabla_\mu\xi=T_0 \tag{A8}$$

we obtain the expression:

$$\lambda=-c^2(\mu_0\rho_0+T_0 s_0).$$

Thus, Eq. (A7) can be rewritten in the form

$$c^{-2}(\mu_0\rho_0+T_0 s_0)u_\mu=-\rho_0\nabla_\mu\alpha-s_0\nabla_\mu\xi. \tag{A9}$$

We note than in the case of isentropic flow Eq. (A9) implies the potential character of the flow[14]

$$\nabla_\mu[\alpha+(s_0/\rho_0)\xi]=-c^{-2}[\mu_0+(s_0/\rho_0)T_0]u_\mu.$$

The expression (A4) allows one to construct the energy-momentum tensor according to Eq. (3). Taking Eqs. (A3), (A5), and (A8) into account, we obtain the known expression[14]

$$T_\nu{}^\mu=c^{-2}(\rho_0\mu_0+s_0T_0)u^\mu u_\nu-\delta_\nu{}^\mu P. \tag{A10}$$

The right-hand side of (A9) formally contains only two arbitrarily functions of α and ξ, whereas the four-velocity u^μ has three arbitrary components. This difficulty can be removed by redefining T_0 in the following manner:

$$T_0=u^\mu(\nabla_\mu\xi+\xi_1\nabla_\mu\xi_2).$$

The equations for the Clebsch variables ξ_1 and ξ_2 are obtained by variations with respect to these variables, and have the form of transport equations:

$$u^\mu\nabla_\mu\xi_1=0, \quad u^\mu\nabla_\mu\xi_2=0.$$

The inclusion of ξ_1 and ξ_2 does not change the form of the final equations (A6) and the expression (A10), but in the right-hand side of (A9) it leads to the redefinition $\nabla_\mu\xi\to\nabla_\mu\xi+\xi_1\nabla_\mu\xi_2$ increasing the number of independent functions describing the components u_μ.

*(*Translator's note*). The relativistic ensemble (9) and the "inverse temperature four-vector" were discussed in detail in the Ph.D. Thesis of M. Kovacich (U. C. Irvine, 1976); cf. also M. Kovacich and M. E. Mayer, Ann. Israel Phys. Soc. **2**, 928 (1978).

[1)]In the presence of a gravitaional field one must take into account the explicit dependence of the invariants (54) on the metric. The flat metric has been defined above.

[2)]In fact, the second term in Eq. (73) is missing In Rothen's paper.

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