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Profile of coherent vortices in two-dimensional turbulence

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The inverse cascade of two-dimensional turbulence in a restricted domain leads to creating a coherent flow containing a number of system-size vortices. We examine the case of forcing turbulence with zero bottom friction where the final statistically steady state is determined by viscosity. We analytically establish structure of the coherent vortices in the state.

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In the theory of turbulence, one of the central problem is to understand and to describe phenomena related to interaction of the mean (coherent) flow and of the flow fluctuations. There are different situations depending on concrete conditions. Say, in the three-dimensional developed turbulence fluctuations take energy from the mean velocity exited at the integral scale of turbulence whereas in convection coherent structures are created from fluctuations [1–3]. Here, we consider the forced two-dimensional (2D) turbulence excited by pumping acting at a scale smaller that the box size [4]. Then at some conditions large-scale coherent structures, including big vortices, are generated from small-scale fluctuations. This process occurs because the 2D Navier–Stokes equation favors energy transfer to larger scales, the phenomenon is known as the inverse energy cascade [5–7].

We analyze the 2D case where turbulence is excited by pumping force correlated at a scale l_p much smaller than the box size L. The inverse energy cascade carries the energy produced by pumping to larger and larger scales. In an infinite system the inverse cascade is spread up to a scale l_α where the bottom friction terminates the inverse cascade [5–7]. If the box size L is smaller than l_α then the energy carried by the inverse cascade is accumulated at the box size, that gives rise to creating the coherent structure. Already the first experiments on 2D turbulence [8] have shown that in a vessel with relatively small bottom friction, the energy accumulation leads to formation of coherent flow containing vortices with sizes of the order of the vessel size. The vortices are separated by regions with a hyperbolic flow.

Subsequent direct numerical simulations [9] and experiments [10] demonstrated that these coherent vortices have the well-defined mean flow profiles with a power-law radial dependence of the mean vorticity Ω in the inner region of the vortex. The flow profile in the region depends neither on the boundary conditions nor on the type of forcing. In Refs. [11] results of extensive simulations of 2D turbulence in a periodic box were presented together with theoretical explanation of the results. They show that the internal vortex structure is universal: the vortex flow is isotropic and outside a small core the mean vortex (polar) velocity U is independent of the separation from the vortex center. The value of the mean velocity is $U = (3\epsilon/\alpha)^{1/2}$ where ϵ is the energy pumping rate per unit mass and α is the bottom friction coefficient. Therefore the mean vorticity behaves as $\Omega \propto r^{-1}$ where r is separation from the vortex center. Fluctuations of the flow in the regime are smaller than the mean flow profile.

Here we analytically examine the case where the bottom friction is negligible that is the limit $\alpha \to 0$. Then the only dissipative mechanism is viscosity that cannot immediately stop the energy accumulation at the box size, since the viscosity effectiveness diminishes as the scale grows. That leads initially to formation of the same universal structure as in the presence of the bottom friction, but with gradually increasing velocity profile [9]. Then in the inner region of the coherent vortex the mean velocity is $U = (6\epsilon t)^{1/2}$ where t is the pumping duration. Due to the increase of the mean flow viscosity ultimately comes into game stabilizing the coherent flow. We will be interested just in this final statistically homogeneous state caused by viscosity. We assume that the state inherits such properties of the previous stage such as isotropy of the coherent vortices and relative smallness of fluctuations.

The starting point for our theoretical analysis is the forced 2D Navier–Stokes equation for the 2D velocity field $\mathbf{v}$,

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}. \qquad (1)$$

Here p is pressure (per unit mass), ν is the kinematic viscosity coefficient, and $\mathbf{f}$ is an external force (per unit mass) exciting the turbulence. The Navier–Stokes equation (1) has to be supplemented by the incompressibility condition $\nabla\cdot\mathbf{v}=\mathbf{0}$. The pumping force $\mathbf{f}$ is also assumed to be divergentless, $\nabla\cdot\mathbf{f}=\mathbf{0}$. Then the pressure p is determined by the equation

$$\nabla^2 p = -\partial_\alpha v_\beta \partial_\beta v_\alpha, \tag{2}$$

relating the pressure to the velocity gradients.

The pumping force $\mathbf{f}$ is assumed to be a random function of time and space with homogeneous statistics, with a correlation length l_p much less than the system size L. The energy production rate per unit mass ϵ is then expressed as $\epsilon=\langle\mathbf{f}\cdot\mathbf{v}\rangle$. Here and below angular brackets mean temporal averaging. For a short-correlated in time forcing the average $\langle\mathbf{f}\cdot\mathbf{v}\rangle$ is independent of the velocity profile. The inverse cascade and, further, the coherent flow are formed provided the Reynolds number is large. Estimating the flow velocity at the pumping length as $v_p\sim(\epsilon l_p)^{1/3}$ we find that the Reynolds number $v_p l_p/\nu$ is large provided $l_p\gg l_\nu$ where l_ν is the viscous length

$$l_\nu = \nu^{3/4}/\epsilon^{1/4}. \tag{3}$$

The inequality $l_p\gg l_\nu$ is assumed to be satisfied below.

We decompose the flow velocity into the mean (coherent) part $\mathbf{V}$ and the fluctuation one that we designate as $\mathbf{v}$. In the final statistically steady state caused by viscosity fluctuations $\mathbf{v}$ are small in comparison with the mean velocity $\mathbf{V}$ and the main energy dissipation is related to the mean velocity $\mathbf{V}$. Estimating the velocity gradient as V/L we find the energy dissipation rate per unit mass $\nu V^2/L^2$. Balancing the dissipation rate by the energy pumping rate ϵ one obtains

$$V \sim L(\epsilon/\nu)^{1/2}. \tag{4}$$

The estimation is correct outside the coherent vortices, in the hyperbolic region.

The mean (coherent) flow produces stretching the flow fluctuations making their life time of the order of $|\nabla V|^{-1}$ where $|\nabla V|$ is the characteristic velocity gradient of coherent flow. That leads to the estimation

$$v^2 \sim \frac{\epsilon}{|\nabla V|}, \tag{5}$$

for typical velocity fluctuations. The estimation (5) implies that the fluctuation correlation length is less than the characteristic scale of the coherent motion. Note that due to the assumed inequality $l_p\gg l_\nu$ viscosity does play no role in forming the fluctuation life time and the estimation (5) can be obtained simply by dimension reasoning. It follows from Eqs. (4), (5) that in the hyperbolic region

$$v^2 \sim \sqrt{\epsilon\nu} \sim V^2 l_\nu^2/L^2. \tag{6}$$

Thus, the inequality $l_\nu\ll L$ guarantees smallness of the velocity fluctuations in comparison with the mean velocity there.

We now pass to analyzing the flow characteristics inside the vortex. The main goal of the analysis is to establish the profile of the mean flow there. As in the case of finite bottom friction [10, 11], the vortex interior can be separated into the vortex core and the region outside the core where the average velocity profile reveals universal scaling properties. To describe the vortex, we introduce polar coordinates in the reference system with the origin at the vortex center: r is the distance from the vortex center and φ is the corresponding polar angle. An isotropic vortex can be described in terms of the mean polar velocity U, which is a function solely of r. Then the mean vorticity $\Omega=U/r+\partial_r U$ is a function solely of r as well. We designate the radial component of the fluctuating velocity as v and its polar component as u.

One argued in Ref. [11], that correlation functions of u,v odd in v are much smaller than ones even in v. It is explained by negligible role of viscosity in forming relevant fluctuations and by smallness of the friction coefficient α in comparison with characteristic mean velocity gradients. Then the time reversibility is weakly broken and that is why the correlation functions of u,v odd in v are small: if the system is invariant under $t\to -t$ then the correlation functions have to be zero. In our case where the system state is determined by viscosity, the time reversibility is strongly broken. Therefore we expect that all correlation functions of u,v, odd and even in v, are equally estimated.

We now start deriving equations for the mean velocity. Averaging the Navier–Stokes equation (1) and taking its radial component, one obtains

$$\partial_r\langle r v^2\rangle + r\partial_r\langle p\rangle = U^2 + \langle u^2\rangle. \tag{7}$$

In deriving the equation (7) we exploited the time homogeneity, the vortex isotropy and the incompressibility condition $\partial_\varphi u+\partial_r(rv)=0$. Note that the viscosity is dropped from Eq. (7). Taking the polar component of the averaged equation (1), one finds

$$\frac{1}{r^2}\partial_r\left(r^2\langle uv\rangle\right) - \nu\left[\frac{1}{r}\partial_r(r\partial_r U) - \frac{U}{r^2}\right] = 0, \tag{8}$$

where, again, we exploited the vortex isotropy and the flow incompressibility.

We now turn to the energy balance. By taking a scalar product of $\mathbf{v}$ with the Navier–Stokes equation (1) and averaging the result, one gets the energy budget

$$\frac{1}{r}\partial_r\left[rU\langle uv\rangle + r\left\langle v\left(\frac{u^2+v^2}{2}+p\right)\right\rangle\right] - \\ -\nu U\left[\frac{1}{r}\partial_r(r\partial_r U)-\frac{U}{r^2}\right]=\epsilon. \quad (9)$$

The equation reads that inside the vortex the energy produced by the pumping source (right-hand side of the equation) is carried in space (from the vortex, as we will see below) due to fluctuations (the first term in the left-hand side) and is partly dissipated (the second term in the left-hand side).

We consider the case where fluctuations are much smaller than the average flow. Then it is possible to neglect in Eq. (9) the third order term over fluctuations. We obtain from Eq. (2) an equation for the only possibly relevant contribution to the pressure

$$\nabla^2 p = -2\frac{\partial_r U}{r}\partial_\varphi v + \frac{2U}{r}\partial_r u. \quad (10)$$

Here ∇^2 can be estimated as l_c^{-2} where l_c is the correlation length of the velocity fluctuations. If l_c is much smaller than the radius r (that is the characteristic length of the mean velocity variations) then the pressure (10) is small and the term with pressure can be dropped from Eq. (9). We postpone justifying the inequality $l_c \ll r$ to the end of the paper. Assuming it, we arrive at the equation

$$\frac{1}{r}\partial_r\left(rU\langle uv\rangle\right) - \nu U\left[\frac{1}{r}\partial_r(r\partial_r U)-\frac{U}{r^2}\right]=\epsilon, \quad (11)$$

reflecting the energy budget.

Inside the vortex, at $r \ll L$, the viscous terms in Eqs. (8), (11) appear to be negligible. Therefore we obtain from the equation (8) that $\langle uv\rangle \propto r^{-2}$, that is

$$\langle uv\rangle = C\sqrt{\epsilon\nu}\,\frac{L^2}{r^2}, \quad (12)$$

where $C \sim 1$ in accordance with Eq. (6). Next, we obtain from Eq. (11)

$$U = \frac{\Omega_0 r}{2} + \frac{\epsilon^{1/2}}{\nu^{1/2}}\frac{r^3}{2CL^2}, \quad (13)$$

where the first contribution, corresponding to the rigid body rotation, is zero mode of the operator in the left-hand side of Eq. (11). An estimate of the constant Ω_0, based on Eq. (4), is $\Omega_0 \sim (\epsilon/\nu)^{1/2}$. Thus, we end up with the regular Taylor expansion for U, the radius of the Taylor series is estimated as L. One can check that the viscous terms in Eqs. (8), (11) produce contributions to U and $\langle uv\rangle$ of higher order in r/L than those kept in Eqs. (12), (13).

We conclude that the viscous energy dissipation inside the vortex is small. Therefore the energy produced there by the pumping force is carried (due to the velocity fluctuations) into the hyperbolic region where it is dissipated. Note that the direction of the energy flux inside the vortex in our (purely viscous) case is opposite to one realized in the case of finite bottom friction [11], where the energy is carried inside the vortex.

The expression (12) for the velocity fluctuations demonstrates that the fluctuations grow as r diminishes. It is explained by the fact that the main contribution to the velocity (13) corresponds to the rigid body rotation that does not influence fluctuations. Thus solely the second term in the expression (13) is responsible for the fluctuation suppression due to their stretching by the mean flow. Therefore, to estimate fluctuations, one should substitute $\nabla V \sim \sqrt{\epsilon/\nu}\,r^2/L^2$ into (5). Then one obtains the behavior (12), indeed. Since the main contribution to the velocity inside the vortex is related to the rigid body rotation then the main contribution to the average pressure inside the vortex in accordance with Eq. (7) is $\langle p\rangle = \mathrm{const} + \Omega_0^2 r^2/8$. Corrections to the expression are proportional to r^4.

Now we should check our assumption that the correlation length of the velocity fluctuations l_c is much smaller that the characteristic mean velocity scale, for the vortex this scale is r. First of all, in the hyperbolic region the stretching rate V/L is much larger than the inverse non-linear time $\epsilon^{1/3}/l_p^{2/3}$ as it follows from the inequality $l_p \gg l_\nu$. Therefore the velocity fluctuations are in the passive regime and their correlation length is equal to l_p. This quantity is assumed to be much smaller than the scale L of the mean flow, indeed. If $l_p^4 \ll L^3 l_\nu$ then the passive regime is valid in the interval of scales $(l_\nu/l_p)^{1/3}L < r < L$ inside the vortex. In this interval $l_p \ll r$ that is the inequality, needed for us, is satisfied. For scales $r < (l_\nu/l_p)^{1/3}L$ the nonlinearity at the pumping scale becomes stronger than stretching produced by the mean flow. Therefore the traditional inverse cascade occurs in some interval of scales. Besides, l_c remains less than r in the interval $l_\nu^{1/4}L^{3/4} < r < (l_\nu/l_p)^{1/3}L$. At smaller radii our scheme based on the equations (8), (11) doesn't work. In the region $r < l_\nu^{1/4}L^{3/4}$, that can be called the vortex core, the fluctuations dominate. One can say, that in the vortex core the traditional inverse cascade is realized (on the background of the rigid body rotation). If $l_p^4 \gg L^3 l_\nu$ then the passive regime is realized down to scales $r \sim l_p$.

In this case the vortex core radius is equal to the pumping length. An analysis of the flow behavior inside the core needs a separate analysis that is outside the scope of this work.

To conclude, we examined turbulence excited by small-scale pumping in finite two-dimensional box with vanishing bottom friction. At large Reynolds numbers the coherent large-scale flow is formed dominating over fluctuations. The coherent flow contains some vortices where rigid body rotation is realized. The flow fluctuations grow to the center of the vortex and dominate in the narrow region near the vortex center that can be called the vortex core. Outside the vortices a hyperbolic flow is formed where the coherent flow dominates everywhere. Our analysis is applicable to two-dimensional flows without bottom friction, say, to the soap film turbulence [12].

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1. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, MIT Press (1971), v. I; Diver (2007), v. II.
2. A. A. Townsend, *The Structure of Turbulent Shear Flow*, Cambridge University Press, Cambridge (1976).
3. D. Lohse and K.-Q. Xia, Annu. Rev. Fluid Mech. **42**, 335 (2010).
4. G. Boffetta and R. E. Ecke, Annu. Rev. Fluid Mech. **44**, 427 (2012).
5. R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
6. C. E. Leith, Phys. Fluids, **11**, 671 (1968).
7. G. K. Batchelor, Phys. Fluids **12**, 233 (1969).
8. J. Sommeria, J. Fluid Mech. **170**, 139 (1986).
9. M. Chertkov, C. Connaughton, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett **99**, 084501 (2007).
10. H. Xia, M. Shats, and G. Falkovich, Phys. Fluids **21**, 125101 (2009).
11. J. Laurie, G. Boffetta, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **113**, 254503 (2014).
12. H. Kellay and W. I. Goldburg, Repts. Prog. Phys. **65**, 845 (2002).