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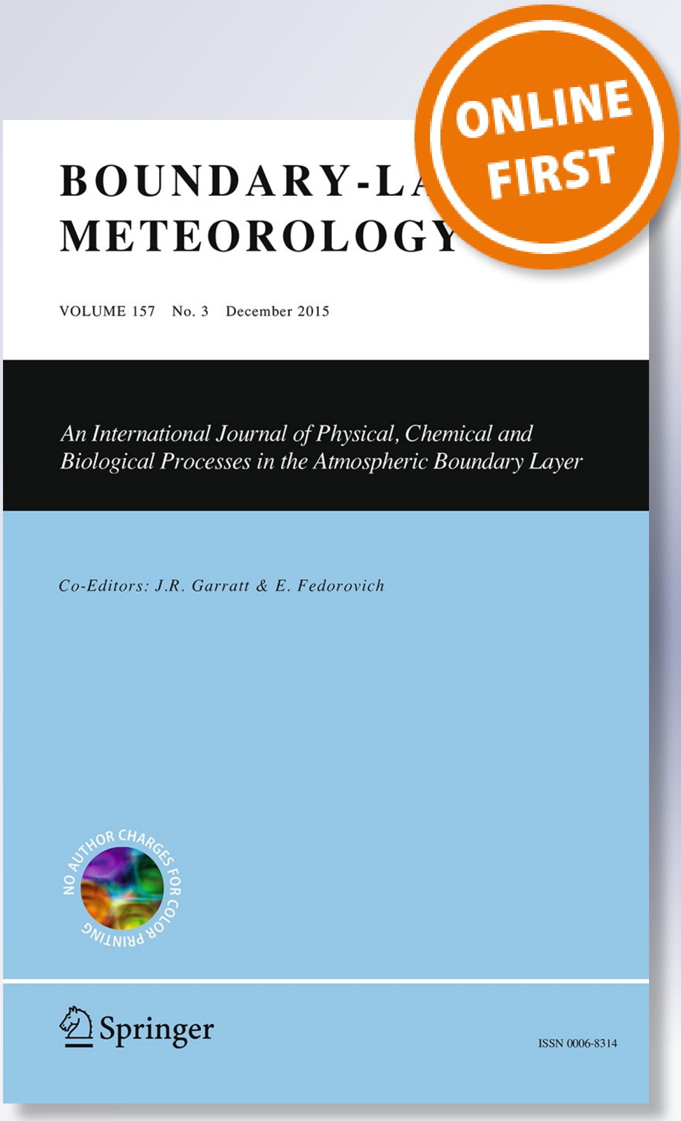

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ARTICLE

Particle Dispersion in the Neutral Atmospheric Surface Layer

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Abstract We address theoretically the longstanding problem of particle dispersion in the lower atmosphere. The evolution of particle concentration under an absorbing boundary condition at the ground is described. We derive a close-form solution for the downwind surface density of deposited particles and find how the number of airborne particles decreases with time. The problem of the plume formation above the extended surface source is also solved analytically. At the end, we show how turbophoresis modifies the mean settling velocity of particles.

Keywords Analytical solution · Dispersion theory · Heavy particles · Settling velocity

1 Introduction

The description of the spreading of a particle cloud in the lower atmosphere is of great importance for a variety of disciplines, from meteorology and urban planning to botany. Depending on the physical context, the term "particles" may refer to dust, sand, droplets, snow, aerosols, seeds, spores, pollen, etc. The particle motion in the air is determined by an interplay of turbulent diffusion, gravitational sedimentation and advection by a mean flow. Even apart from gravity, the situation is highly anisotropic and inhomogeneous, since both the mean wind velocity and the turbulent diffusivity vary with height. Majority of the previous theoretical studies were devoted to the equilibrium concentration profiles in the presence of permanent sources of particles. The goal of the present work is to develop an analytic framework that provides the description of the time evolution of the concentration field. To

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describe statistically the particle transport in a turbulent environment we adopt a standard model of turbulent diffusion, which is based on the concept of eddy diffusivity (Monin and Yaglom 1971). While this approximation is not universally applicable (Corrsin 1974), it allows us to make the complex problem of atmospheric dispersion analytically tractable.

We consider particles, whose density is much larger than the density of air, and each particle is assumed to be so small that the flow around it is viscous. The interaction between particles and their influence on the flow can be neglected as long as the volume fraction of the particle phase is low. Under these assumptions, the equation of particle motion becomes (Maxey and Riley 1983)

$$\frac{d\mathbf{v}}{dt} = -\frac{\mathbf{v} - \mathbf{u}(\mathbf{r}, t)}{\tau} + \mathbf{g} \tag{1}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the particle coordinate and velocity, $\mathbf{u}(\mathbf{r}, t)$ is the fluid velocity field and $\mathbf{g}$ is gravitational acceleration. We start from the case where the particle response time τ is much less than the Kolmogorov time scale of turbulence, so that the role of particle inertia in the process of advection is insignificant. Then, particles follow the turbulent flow of the air, except for the gravitational settling and we pass to the equation

$$\mathbf{v} = \mathbf{u}(\mathbf{r}, t) + \mathbf{g}\tau, \tag{2}$$

which describes the dynamics on time scales $t \gg \tau$. The effects of inertia are discussed later in Sect. 5.

Let us introduce a reference system with the z-axis perpendicular to the ground (that is, parallel to the field of gravity $\mathbf{g}$). The x-axis is along the horizontal direction of the mean wind. The particle motion in the crosswind direction is beyond the scope of this work, thus our consideration is effectively two-dimensional. Treating the fluctuating part of the incompressible velocity field $\mathbf{u}(\mathbf{r}, t)$ as short-correlated in time, it is straightforward to derive from (2) the Fokker–Plank equation for the particle probability density. The same equation describes the particle concentration field $\theta(x, z, t)$ if we consider distribution of a large number of particles in space,

$$\partial_t \theta = \partial_z [D(z)\partial_z \theta] + g\tau \partial_z \theta - u(z)\partial_x \theta, \tag{3}$$

see e.g. Okubo and Levin (1989), Falkovich et al. (2001). Here $D(z)$ is the vertical turbulent diffusivity and $u(z)$ is the mean horizontal velocity component of the wind. Our model implies that turbulence is homogeneous in the horizontal direction. The molecular diffusion is assumed to be unimportant in comparison with that due to turbulent motion of air, and we also neglect the horizontal turbulent diffusion in comparison with the mean advection.

In a fully developed turbulent boundary layer, the kinetic energy of fluctuations is approximately independent of the distance from the surface z, while the typical scale of turbulent eddies grows as z. This gives the vertical turbulent diffusivity in the atmospheric surface layer linearly growing with the height z,

$$D(z) = \mu z, \tag{4}$$

where $\mu = \kappa u_*$ with the friction velocity u_* and the von Karman constant κ.

Following the usual practice, we approximate the mean horizontal velocity profile by the power law

$$u(z) = \beta z^m, \tag{5}$$

in which $\beta = u_* C_p / z_*^m$ and z_* is the aerodynamic roughness length. For neutral atmospheric conditions, the values $m = 1/7$ and $C_p = 6$ are usually adopted Brutsaert (1982). In the limit $m \to 0$ one passes to the logarithmic profile

$$u(z) = \frac{u_*}{\kappa} \ln \frac{z + z_*}{z_*}. \tag{6}$$

The steady-state solutions of Eq. 3 with the height-dependent coefficients given by (4) and (5) have been previously found for the particle dispersion from line (Okubo and Levin 1989; Rounds 1955; Godson 1958) or area sources (Chamecki and Meneveau 2011; Pan et al. 2013). The non-stationary dispersion processes described by these equations are less understood, and we here focus on non-stationary concentration profiles. We consider particle spread from a surface source, which starts acting at some moment, as well as a source-free evolution of an initial cloud. At the first step, we exclude the effects of horizontal advection by passing to the reduced description in terms of the density integrated over the horizontal plane $\tilde{\theta}(z,t) = \int_{-\infty}^{+\infty} \theta(x,z,t)\mathrm{d}x$. This vertical profile of the particle concentration obeys the following closed equation,

$$\partial_t \tilde{\theta} = \partial_z [D(z)\partial_z \tilde{\theta}] + g\tau \partial_z \tilde{\theta}. \tag{7}$$

Thus, the description of the vertical distribution is decoupled from the problem of the horizontal transport. While the complete distribution $\theta(x,z,t)$ cannot be recovered from the integrated one $\tilde{\theta}(z,t)$, the knowledge of $\tilde{\theta}$ allows us to calculate the x-coordinate of the centre-of-mass of the particle cloud as a function of time.

Equations 3 and 7 have to be supplemented by boundary conditions at the ground level. Since the coefficient near the second-order derivative in these equations vanishes at $z = 0$, then we should impose a boundary condition at some reference height r where the diffusivity is still finite. Technically, it is sometimes convenient to pose the condition at $z = 0$ shifting $z \to z + r$ in the diffusion coefficient (4): $D(z) \to \mu(z + r) \equiv \mu z + D_0$, where $D_0 = \mu r$. Note that in the general case of a rough surface, the effective diffusivity could be modelled as having some non-zero value D_0 at the boundary (Smith 2008). The type of boundary condition depends on the particular physical situation; details are given below for different cases.

2 Passive Scalar

The ratio of the settling velocity in still air to the turbulent velocity, $\gamma = g\tau/\mu$, is a dimensionless measure of the relative importance of gravitational settling and turbulent dispersion. If $\gamma \ll 1$, then gravity in Eq. 7 can be neglected and we pass to

$$\partial_t \tilde{\theta} = \partial_z [D(z)\partial_z \tilde{\theta}], \tag{8}$$

which is applicable to vertical transport of vapour (Sutton 1943; Frost 1946) and fine particles (Chamberlain 1967); an airborne impurity with negligible inertia is called passive scalar.

Our goal is to describe the time evolution of a dispersing cloud, obtained from the sudden release of passive scalar into the atmosphere at height z_0 above the ground. Then, the initial condition is $\tilde{\theta}(z,0) = N_0\delta(z - z_0)$ where N_0 is the number of particles. Firstly, let us consider the situation where the surface acts as perfect reflector so that the vertical particle flux is zero at ground level: $[D(z)\partial_z \tilde{\theta}]_{z=r} = 0$. The corresponding boundary-value problem for Eq. 8 has been considered by many authors, see Monin and Yaglom (1971) and references therein. At $r \to 0$ it is possible to construct the following solution

$$\tilde{\theta}(z,t) = \frac{N_0}{\mu t} \exp\left(-\frac{z + z_0}{\mu t}\right) \mathcal{I}_0\left(\frac{2\sqrt{z z_0}}{\mu t}\right), \tag{9}$$

in which $\mathcal{I}_0$ denotes the modified Bessel functions of the first kind. This concentration profile is shown in the top left panel of Fig. 2. Note that the total amount of airborne particles $\int_0^\infty \tilde{\theta}(z,t)\mathrm{d}z = N_0$ is time independent. The property is related to zero particle flux at $z = 0$.

From Eq. 8 we readily find that the mean vertical coordinate $\langle z(t)\rangle = N_0^{-1}\int_0^\infty z\,\tilde{\theta}(z,t)\mathrm{d}z$ growths linearly with time

$$\langle z(t)\rangle = z_0 + \mu t. \tag{10}$$

This result can also be derived directly from Eq. 8, see e.g. Batchelor (1964), Chatwin (1968). On the time scales $t \gg z_0/\mu$ the system loses information about details of the initial distribution and one can substitute $z_0 = 0$ to obtain from Eq. 9 the following universal long-term asymptotics,

$$\tilde{\theta}(z, t \gg z_0/\mu) \approx \frac{N_0}{\mu t}\exp\left(-\frac{z}{\mu t}\right). \tag{11}$$

The complete distribution $\theta(x,z,t)$ cannot be recovered from the integrated one $\tilde{\theta}(z,t)$. Nevertheless, it is possible to extract some information concerning the particle motion along the ground. Let us assume that the particles are released initially at $x = 0$, $z = z_0$. The typical distance travelled in the downwind direction up to time t is given by the statistical average $\langle x(t)\rangle = N_0^{-1}\int_{-\infty}^{+\infty}\int_0^\infty x\theta(x,z,t)\mathrm{d}x\mathrm{d}z$, where θ is the solution of Eq. 3 with the initial condition $\theta(x,z,0) = \delta(x)\delta(z - z_0)$. One derives from Eq. 3 the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle = \langle u(z(t))\rangle, \tag{12}$$

in which $\langle u(z(t))\rangle = N_0^{-1}\int_0^{+\infty}\mathrm{d}z\, u(z)\tilde{\theta}(z,t)$. Using Eqs. 5 and 11 we find the solution

$$\langle x(t)\rangle = \frac{\Gamma(m+1)}{m+1}\beta\mu^m t^{m+1}. \tag{13}$$

It is noteworthy that approximately the same result follows from the naive relation $\langle x(t)\rangle \approx \int_0^t u(\langle z(t')\rangle)\mathrm{d}t' = \beta\mu^m t^{m+1}/(m+1)$ which is justified by the weakness of the z-dependence of the mean wind velocity. Indeed, in neutral atmospheric conditions (small m) the missing factor $\Gamma(m+1)$ is close to unity: $\Gamma(8/7) \approx 0.94$. Note also that for the logarithmic profile (6), the horizontal displacement grows as follows: $\langle x(t)\rangle \approx (u_* t/\kappa)\ln(\mu t/z_*)$ (Chatwin 1968).

Now let us turn to the case of an absorbing underlying surface. We neglect particle rebound and re-suspension, then the appropriate boundary condition for Eq. 8 is $\tilde{\theta}|_{z=r} = 0$. Equivalently, one can add a constant correction $D_0 = \mu r$ to diffusivity (4) and pass to the equation

$$\partial_t\tilde{\theta} = \mu\partial_z[(z+r)\partial_z\tilde{\theta}], \tag{14}$$

which is supplemented by zero boundary condition at $z = 0$. A non-zero surface diffusivity provides finite particle flux to the ground, which decreases the total number of particles in the atmosphere, provided there are no sources. We wish to find the probability that a particle released initially at z_0 has not been absorbed up to time t. Passing to the variable $\rho = 2\sqrt{(z+r)/\mu}$, one obtains from Eq. 14

$$\partial_t\tilde{\theta} = \frac{1}{\rho}\partial_\rho[\rho\partial_\rho\tilde{\theta}]. \tag{15}$$

Thus we map our problem to the $2d$ isotropic diffusion process. In terms of Eq. 15, the zero boundary condition implies that we deal with an absolutely absorbing cylinder with the radius $\rho_\star = 2\sqrt{r/\mu}$. The problem of two-dimensional random walk near an absorbing cylinder is well known in the theory of trapping reaction and reaction-diffusion systems [see e.g. Havlin et al. (1990)]. For $z_0 \gg r$, the late-time survival probability is estimated as $\ln(z_0/r)/\ln(\mu t/r)$. Therefore, the passive scalar concentration in the atmosphere decreases with time logarithmically, which is very slow. Note that there is no decay in the limit $r \to 0$.

3 Heavy Particles

For sufficiently heavy particles the gravitational sedimentation cannot be ignored and we have to return to the Eqs. 3 and 7. To describe the time evolution of an initial cloud, one should specify the boundary condition at the ground. In general, the surface plays the role of a sink for the airborne concentration field, and it is physically plausible that irrespective of the details of particle-ground interaction, the deposition rate of airborne particles is proportional to the surface concentration (Monin 1959; Calder 1961). Then, equating the particle deposition flux to that obtained from the (3), one defines the so-called radiative boundary condition

$$[D(z)\partial_z\theta + g\tau\theta]_{z=r} = [v_\mathrm{d}\theta]_{z=r}, \tag{16}$$

where v_d is the deposition velocity and r is the height of the reference surface where the flux is measured. The choices $v_\mathrm{d} = 0$ and $v_\mathrm{d} \to \infty$ correspond to the perfectly reflecting and perfectly absorbent boundary conditions, respectively, and in the intermediate case $0 < v_\mathrm{d} < \infty$, one deals with a partially absorbing surface. Here we consider the general situation when deposition velocity is a parameter. For the sake of convenience, we introduce a finite surface diffusivity $D_0 = \mu r$ and impose the boundary condition (16) at $z = 0$, taking the limit $r \to 0$ at the end of calculations (see Appendices 1 and 2).

Let us consider the dispersion of N_0 particles released initially at height z_0 above the ground. A crucial question is how far the particles can be transported by the wind and how the amount of airborne material decrease with time. The number of particles in atmosphere is given by $N(t) = \int_0^\infty \tilde{\theta}(z,t)\mathrm{d}z$, where $\tilde{\theta}(z,t)$ is the solution of Eq. 14 with initial condition $\tilde{\theta}(z,0) = N_0\delta(z - z_0)$ and boundary condition (16). Apparently, the ratio $N(t)/N_0$ can be interpreted as the probability $p(t)$ that the particle, starting at z_0, has not been deposited on the ground up to time t. Using the Laplace transform technique, one obtains the following closed result (see Appendix 1),

$$p(t) = \frac{1}{\Gamma(\gamma)} \int\limits_0^{z_0/\mu t} \xi^{\gamma-1} e^{-\xi}\mathrm{d}\xi, \tag{17}$$

which is illustrated in Fig. 1a. To our knowledge, this simple relation has not been noted previously. From (17), we see that at large time the survival probability decays according to a power law: $p(t) \propto t^{-\gamma}$. The mean deposition time T is given by

$$T = -\int_0^\infty t\,\frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}t = \frac{\Gamma(\gamma-1)}{\Gamma(\gamma)}\frac{z_0}{\mu}, \tag{18}$$

provided that $\gamma > 1$, and is infinite for $\gamma \le 1$.

It should be stressed that the survival probability does not depend on v_d at $r \to 0$. The formal explanation of this result is that for the transport equation (8) with diffusivity (4) the boundary at $z = 0$ is *adhesive*, see e.g. Kampen (1992). For this reason, the effect of any boundary condition is cannot be distinguished from the perfectly absorbing wall: the particles come to rest at $z = 0$ at the rate that is independent on the specific type of

Fig. 1 **a** The time dependence of the survival probability p (see Eq. 17) for a heavy particle released initially at the height z_0 above the ground. **b** The time dependence of the total deposition flux, see Eq. 20

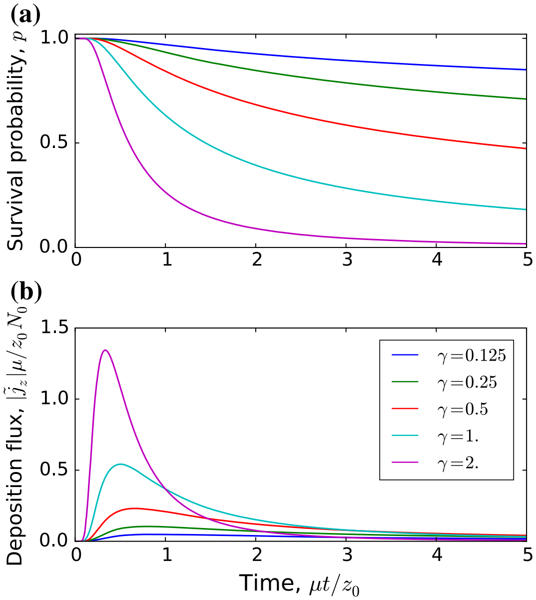
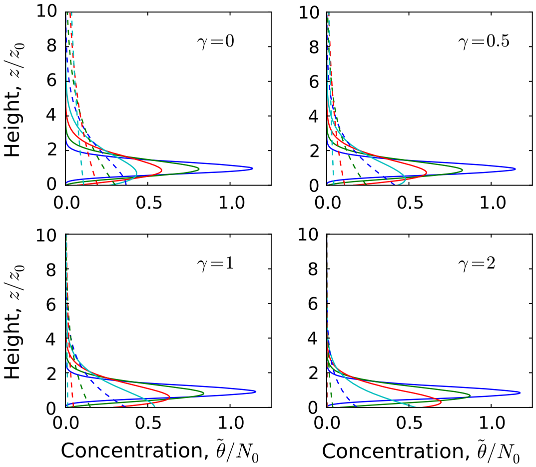


Fig. 2 The vertical concentration profile (see Eq. 19) of the particles released initially at the height z_0 above the ground for different moments of time: $\mu t/z_0 = 2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2$ and 2^3



particle-wall interaction. One can conclude from this that there is a universal solution of the initial-boundary value problem that is valid for any v_{d} as long as $r \to 0$. Indeed, it is easy to verify that the following time-dependent distribution

$$\tilde{\theta}(z,t) = \frac{N_0}{\mu t}\left(\frac{z_0}{z}\right)^{\gamma/2} \exp\left(-\frac{z+z_0}{\mu t}\right) \mathcal{I}_{\gamma}\left(\frac{2\sqrt{zz_0}}{\mu t}\right), \tag{19}$$

is the exact solution of Eq. 8, which reduces to $N_0\delta(z - z_0)$ for $t \to 0$. In Kampen (1992) this relation is derived in the particular case $\gamma = 1$. At $\gamma = 0$ one obtains the well-known solution (9) for passive scalar Monin and Yaglom (1971). The ratio $\tilde{\theta}(z,t)/N_0$ describes the probability density that the particle, released initially at $z = z_0$, is at height z after time t, see Fig. 2. Note that integration of distribution (19) over z gives exactly (17).

Using Eq. 19, we express the total ground deposition flux $\tilde{j}_z = [D(z)\partial_z\tilde{\theta} + g\tau\tilde{\theta}]_{z=0}$ as

$$\tilde{j}_z = -\frac{N_0}{\Gamma(\gamma)}\frac{z_0^\gamma}{\mu^\gamma t^{\gamma+1}}\exp\left(-\frac{z_0}{\mu t}\right), \tag{20}$$

see Fig. 1b. The same expression follows from the relation $\tilde{j}_z = N_0 \mathrm{d}p/\mathrm{d}t$. The intensity of deposition reaches its maximum at time

$$t_m = \frac{z_0}{(\gamma+1)\mu}. \tag{21}$$

Next we describe the motion of the centre-of-mass of the particle cloud whose downwind and vertical coordinates are given by the statistical moments $\langle x(t)\rangle = N^{-1}(t)\int_0^\infty x\ \theta(x,z,t)\mathrm{d}x\mathrm{d}z$ and $\langle z(t)\rangle = N^{-1}(t)\int_0^\infty z\ \tilde{\theta}(z,t)\mathrm{d}z$, respectively. From (3) and (7) one obtains the following evolution equations

$$\partial_t\langle z(t)\rangle = \mu(1-\gamma) - \langle z(t)\rangle\partial_t \ln p(t), \tag{22}$$
$$\partial_t\langle x(t)\rangle = \langle u(z(t))\rangle - \langle x(t)\rangle\partial_t \ln p(t), \tag{23}$$

in which $\langle u(z(t))\rangle = N^{-1}(t)\int_0^\infty u(z)\ \tilde{\theta}(z,t)\mathrm{d}z$. Equation 22 yields exactly

$$\langle z(t)\rangle = \frac{z_0}{p(t)} + \frac{(1-\gamma)\mu}{p(t)}\int_0^t p(t')\mathrm{d}t', \tag{24}$$

while Eq. 22 gives

$$\langle x(t)\rangle = \frac{1}{p(t)}\int_0^t p(t')\langle u(z(t'))\rangle\mathrm{d}t'. \tag{25}$$

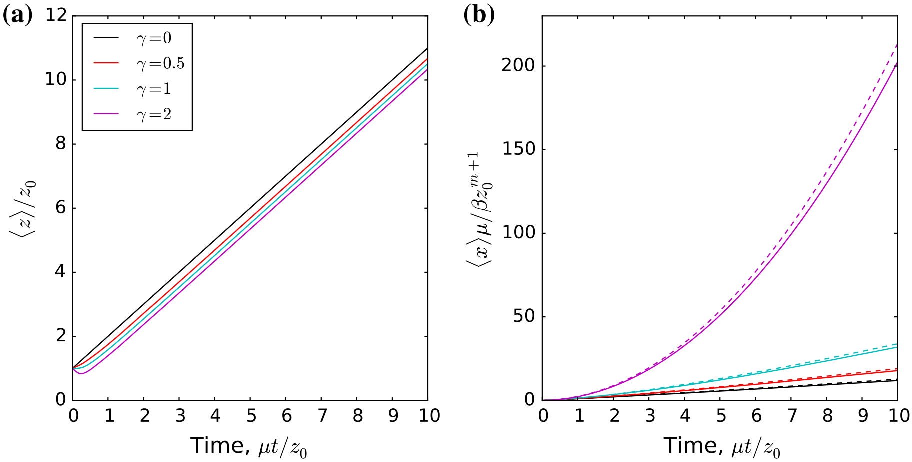


Fig. 3 **a** The vertical coordinate of the centre-of-mass of the particle cloud as a function of time, see Eq. 24. **b** The horizontal coordinate of the centre-of-mass of the particle cloud as a function of time for the power-law profile (5) of wind velocity with $m = 1/7$. The *solid curves* are the exact solutions of Eq. 25 for different values of γ, while the *dotted curves* are the approximate solutions obtained by replacing $\langle u(z(t))\rangle$ by $u(\langle z(t)\rangle)$ in Eq. 25

The vertical coordinate $\langle z(t)\rangle$ of the centre-of-mass grows linearly at large time for any γ, see Fig. 3a. In particular case of passive scalar above the perfectly reflecting ground we have $\gamma = 0$ and $p(t) = 1$ so that Eq. 24 reduces to Eq. 10. The time evolution of the mean $\langle x(t)\rangle$ for different values of γ is shown in Fig. 3b. From Eqs. 5, 17 and 25 one finds that the horizontal coordinate behaves like $\langle x(t)\rangle \propto t^{m+1}$ for $\gamma \le m+1$, whereas $\langle x(t)\rangle \propto t^{\gamma}$ for $\gamma > m+1$.

Finally, let us consider the range of horizontal travel of the particles. The quantity of interest is the resulting surface density $\sigma(x)$ of particles deposited at the downwind distance x from the place where they were released,

$$\sigma(x) = -\int_0^{+\infty} j_z(x, z=0, t)\mathrm{d}t, \tag{26}$$

where $j_z = -D(z)\partial_z\theta - g\tau\theta$ is the vertical particle flux. Solving the corresponding initial-boundary problem for Eq. 3 one obtains (see Appendix 2)

$$\sigma(x) = N_0\left[\frac{(m+1)^2\mu}{\beta z_0^{m+1}}\right]^{-\frac{\gamma}{m+1}} \frac{x^{-\frac{\gamma}{m+1}-1}}{\Gamma\left(\frac{\gamma}{m+1}\right)} \exp\left(-\frac{\beta z_0^{m+1}}{(m+1)^2\mu x}\right). \tag{27}$$

It is straightforward to show that the condition of mass conservation $\int_0^{\infty}\sigma(x)\mathrm{d}x = N_0$ is satisfied, and there are no deposited particles for $x \le 0$ since we neglected horizontal diffusion. One finds from Eq. 27 that the peak of the surface density $\sigma(x)$ is at the downwind distance

$$x_m = \frac{\beta z_0^{m+1}}{(m+1)(\gamma+m+1)\mu} \tag{28}$$

from the place where particles were injected into atmosphere, while the mean horizontal displacement X of the particles is

$$X = \frac{1}{N_0}\int_0^{\infty} x\,\sigma(x)\mathrm{d}x = \frac{\Gamma\left(\frac{\gamma}{m+1}-1\right)}{\Gamma\left(\frac{\gamma}{m+1}\right)} \frac{\beta z_0^{m+1}}{(m+1)^2\mu}, \tag{29}$$

if $\gamma > m+1$, and is infinite otherwise.

The formula (27) has the same structure as the classical result for the ground deposition rate in a steady-state problem when particles are emitted from the permanent line source (Rounds 1955; Godson 1958). Remarkably, in that works the deposition velocity v_d was chosen to be equal to the settling velocity $g\tau$, thus setting the turbulence-induced flux to zero at $z = 0$. Our analysis indicates that the result (27) is universal (i.e. independent on v_d) as long as the reference height r in the boundary condition (16) tends to zero.

4 Surface Source

In the previous sections, the evolution of an initial cloud of particles was considered. Here we treat the case when particles disperse from a surface source that is uniform over a large homogeneous area. Namely, we investigate the evolution of concentration field $\tilde{\theta}$ after the source is switched on at $t = 0$. Then the initial condition is $\tilde{\theta}(z, 0) = 0$.

First, let us consider the situation when the source provides the constant particle concentration near the ground. Equation 7 admits solutions in the self-similar form $\tilde{\theta}(z,t) = t^{-a} f(z/\mu t)$ with some scaling index a, where the unknown function f obeys

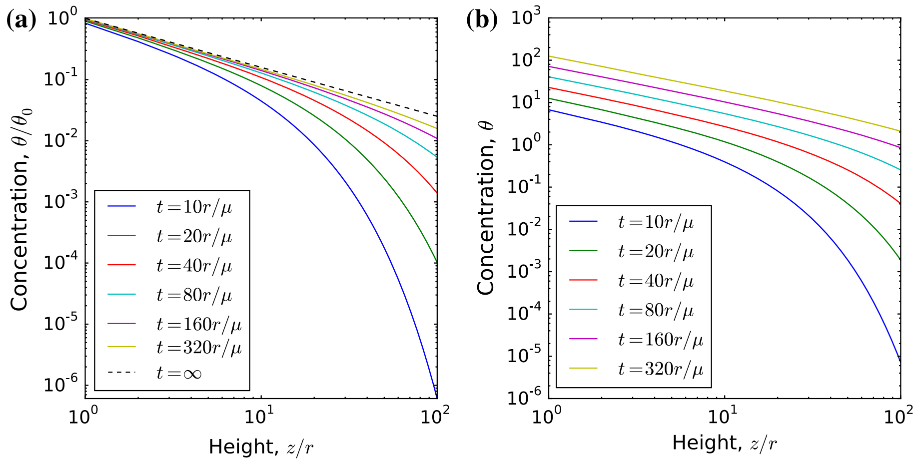


Fig. 4 The vertical concentration profile of the particles with $\gamma = 0.8$ for two types of boundary conditions at the ground: **a** surface source provides constant near-field concentration, see Eq. 31; **b** surface source provides constant injection rate, see Eq. 33

$$\xi \frac{d^2 f}{d\xi^2} + (\xi + \gamma + 1)\frac{df}{d\xi} + af = 0, \tag{30}$$

where $\xi = z/\mu t$. As $\xi \to 0$ in the leading order one obtains $f \propto \xi^{-\gamma}$+constant. Therefore, $\tilde{\theta} \propto t^{-a}[(z/\mu t)^{-\gamma}$+constant] for $z \ll \mu t$. We expect the time independent long-term asymptotic of $\tilde{\theta}$ at small z, which requires one to choose $a = \gamma$. As a result, from Eq. 30 we find

$$\tilde{\theta}(z,t) \propto \frac{1}{z^{\gamma}} \int\limits_{z/\mu t}^{+\infty} d\zeta\, \zeta^{\gamma-1} e^{-\zeta}. \tag{31}$$

This concentration profile cannot be extended all the way to the lower boundary and should be matched to the near-field concentration at $z \sim r$, where the spatial scale r is determined by the vertical size of the source or/and the characteristics of turbulence at ground level.

The self-similar distribution (31) indicates that the equilibrium plume gradually forms above the surface source, maintaining constant concentration of particles near the ground, see Fig. 4a. The plume height grows linearly with time as $\sim\mu t$ and the vertical profile of particle concentration at $z \ll \mu t$ is given by a power law,

$$\tilde{\theta} \propto \frac{1}{z^{\gamma}}. \tag{32}$$

Let us now consider another set-up, where instead of dynamical equilibrium near ground we have a fixed upward flux. This is the case, for instance, when an industry area acts as a permanent sources of pollution at the ground. Then the character of the solution is essentially changed. The self-similarity is still the same, however, one should put $a = 0$ to insure the constant particle flux at the ground. This leads to the following concentration profile

$$\tilde{\theta}(z,t) \propto \int\limits_{z/\mu t}^{+\infty} d\zeta\, \zeta^{-\gamma-1} e^{-\zeta}, \tag{33}$$

which is valid at $z \gtrsim r$.

We conclude from (33) that the dust front propagates to large z with the velocity μ, as in (31). However, instead of the stationary profile (32) at $z \ll \mu t$ we observe dust accumulation (Fig. 4b)

$$\tilde{\theta} \propto \left(\frac{t}{z}\right)^{\gamma}. \tag{34}$$

The more detailed analysis presented in Appendix 3 confirms these results.

5 Effect of Particle Inertia

Equation 3 is derived from the equation of motion (2), which implies that the only effect of particle inertia is the gravitational settling with the terminal velocity $g\tau$. Previous studies have shown that spatially homogeneous turbulence usually increases the average settling velocity due to tendency of inertial particle to accumulate in regions of high flow strain rate or low flow vorticity (see e.g. Wang and Maxey 1993). Here we discuss another mechanism of settling velocity modification—turbophoresis, which is specific for inertial particles in spatially non-uniform random flows.

To take into account the particle inertial response to turbulent fluctuations one should turn to Eq. 2. Let us model incompressible chaotic flow $\mathbf{u}(\mathbf{r}, t)$ by using an Ornstein–Uhlenbeck process having finite correlation time τ_c. At $t \gg \tau, \tau_c$ the vertical distribution $\tilde{\theta}(z, t)$ of particles carried by this flow is described by (Belan 2016; Belan et al. 2014)

$$\partial_t \tilde{\theta} = \frac{St}{1+St}\partial_z^2[D(z)\tilde{\theta}] + \frac{1}{1+St}\partial_z[D(z)\partial_z\tilde{\theta}] + g\tau\partial_z\tilde{\theta}, \tag{35}$$

where the Stokes number is defined as $St = \tau/\tau_c$ and the effective turbulent diffusivity is $D(z) = \int_{-\infty}^{0}\langle u_z(z,t)u_z(z,0)\rangle \mathrm{d}t$, the angle brackets denoting averaging over statistics of the random flow. This equation implies that characteristic time scales τ_c and τ of, respectively, fluid and particle velocities are much smaller than the typical time of evolution of the particle concentration. The structure of Eq. 35 is quite transparent: it has the form of a conservation law $\partial_t\tilde{\theta} = -\partial_z \tilde{j}_z$ for $\tilde{\theta}$, where the vertical particle flux is given by

$$\tilde{j}_z = -D(z)\partial_z\tilde{\theta} - \frac{St}{1+St}\frac{\mathrm{d}D(z)}{\mathrm{d}z}\tilde{\theta} - g\tau\tilde{\theta} \tag{36}$$

The second term in the right-hand side is proportional to the diffusivity gradient and can be interpreted as turbophoretic drift in the direction where the turbulence diffusivity decreases. When diffusivity is linear, $D(z) = \mu z$, the gradient is constant and the equation takes the form (7),

$$\partial_t\tilde{\theta} = \mu\partial_z\left(z\partial_z\tilde{\theta}\right) + \bar{v}\partial_z\tilde{\theta}\,, \tag{37}$$

where the mean settling velocity is enhanced by turbophoresis,

$$\bar{v} = g\tau + \mu St/(1+St). \tag{38}$$

The phenomenon is more pronounced for highly inertial particles. Note in passing that it is straightforward to write the general steady-state solution of (37): $\tilde{\theta}(z) = -\tilde{j}_z/\bar{v} + Az^{-\bar{v}/\mu}$ and self-similar regimes of propagation, like, for instance, $\tilde{\theta}(z,t) = t^{-1}f(z/\mu t)$, with $f(\xi) \to \xi^{-\bar{v}/\mu}$ at $\xi \to 0$ and $f(\xi) \to e^{-\xi}$ at $\xi \to \infty$, which is normalizable for $\bar{v} < \mu$.

Clearly, the random flow described above is a gross simplification of real turbulence, which contains hierarchy of spatial scales of motion associated with a range of time scales.

For this reason, the fluctuating part of the fluid velocity cannot be quantitatively described as a random process with a single correlation time. Nevertheless, our simple model can be used to extract a theoretical estimate of an inertia-induced correction to the settling velocity. Within the surface layer the typical spatial scale of turbulent eddies, which mainly contribute the diffusivity at height z, is proportional to z. The corresponding correlation time is just the turnover time $\tau_c(z) \sim z/u_*$. Thus, it follows from (38) that inertia increases the settling velocity by the quantity of order $u_* St(z)/(1 + St(z))$, where the height-dependent Stokes number is estimated as $St(z) \sim u_*\tau/z$. This increase is more significant closer to the ground.

The role of inertia in the modification of the settling velocity of particles located within the atmospheric surface layer was investigated numerically in Chamecki and Meneveau (2011). For $\mu = 0.32\ \mathrm{ms}^{-1}$ and $\gamma = 0.625$, they reported the 17 % increase in the settling velocity at height $z = 3$ m. However, as explained in the recent work (Pan et al. 2013), that analysis strongly overestimates the effects of inertia due to the incorrect calculation of fluid acceleration in the numerical scheme. In agreement with this remark, our model gives only $\approx$0.5 % correction to the mean vertical velocity for the same set of parameters.

6 Conclusion

We have examined the dispersion of airborne particles under the action of a non-uniform turbulent vertical diffusion, gravitational sedimentation and mean horizontal advection. There is a long-standing interest in this problem in the fields of atmospheric pollution, meteorology, agriculture, biology and many others. Our simple model of statistical description is based on the two-dimensional transport equation (3): only distributions along the vertical and streamwise directions are studied. In contrast to the majority of previous works on the subject, we have focused on non-stationary phenomena. The main part of our analytical results concerns the time evolution of airborne concentration in situations when particles are initially injected into the atmosphere at a finite height above the ground level. In Sect. 2, we analyzed the limit of negligible gravitational sedimentation. If the underlying surface is perfectly reflecting, the vertical profile of particle distribution is given by Eq. 9, while the mean horizontal displacement of particles changes in time according to Eq. 13. For the opposite case of a perfectly absorbent boundary condition, we demonstrate that total amount of airborne material decrease with time logarithmically slow. In Sect. 3, we treated the more general situation when gravity is non-negligible. Our theory predicts the time dependence of the total number of airborne particles (20) and resulting surface density of deposited material (27) in the case of radiative boundary condition (16) at ground level. Moreover, we described the motion of the centre of mass of a particle cloud, see Eqs. 22 and 23. In Sect. 4, we considered the non-stationary distribution of airborne particles above a uniform area source. The sources providing constant near-field concentration and constant particle flux were analyzed separately.

The next step was to take into account particle inertia, which makes the mean vertical drift different from Stokes terminal velocity $g\tau$. We argued that there is an increase in settling velocity because of the turbophoretic drift of inertial particles in the direction of lower turbulence diffusivity, i.e. towards the ground. For sufficiently inertial particles, the factor $g\tau$ should be replaced by some effective settling velocity which accounts for the inertial corrections to mean vertical drift. Note that the turbophoresis-induced correction is height-dependent and may be significant only in close vicinity to the ground.

Finally, let us mention a few issues that are of interest in investigating further. In this study the absorbing boundary condition at the ground level was imposed to describe the particle-ground interaction. We assumed that every particle that comes to rest at the ground remains there. A natural extension includes the effects of particle re-suspension. The corresponding initial-boundary value problem for airborne concentration needs separate analysis, which will be the subject of future work. It should also be noted, that our simple two-dimensional model cannot predict details of particle spreading in crosswind direction which arises from the horizontal turbulent diffusion. In fact, the only mechanism of horizontal transport that was taken into account herein is the advection by the mean flow. Our methods admit generalization for the case of non-zero horizontal diffusivity.

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Appendix 1

Let us consider the dispersion of N_0 particles released initially at the height z_0 above the ground level. Our goal is to derive the number $N(t) = \int_0^\infty \tilde{\theta}(\theta, t)\mathrm{d}z$ of particles in the air as a function of time. To be able to impose the boundary condition at the surface, we regularize the problem by having a non-zero diffusivity at $z = 0$. This give the transport equation

$$\partial_t \tilde{\theta} = \mu \partial_z[(z + r)\partial_z \tilde{\theta}] + g\tau \partial_z \tilde{\theta}, \tag{39}$$

subject to the conditions $[\mu(z+r)\partial_z\tilde{\theta}+g\tau\tilde{\theta}]_{z=0} = [v_\mathrm{d}\theta]_{z=0}$ and $\tilde{\theta}(z, 0) = N_0\delta(z-z_0)$. The spatial scale r is a regularization parameter. At the end of the calculations, the limit $r \to 0$ will be taken.

Performing the Laplace transform

$$\tilde{\theta}_\mathrm{s}(z) = \int_0^{+\infty} \mathrm{d}t e^{-st}\tilde{\theta}(z, t), \tag{40}$$

one obtains an ordinary second-order differential equation

$$\mu\partial_z[(z + r)\partial_z\tilde{\theta}_\mathrm{s}] + g\tau\partial_z\tilde{\theta}_\mathrm{s} - s\tilde{\theta}_s = -N_0\delta(z - z_0), \tag{41}$$

which should be supplemented by the condition $[D(z)\partial_z\tilde{\theta}_\mathrm{s} + g\tau\tilde{\theta}_\mathrm{s}]_{z=0} = [v_\mathrm{d}\theta_\mathrm{s}]_{z=0}$.

We pass to the new variable $\xi = \sqrt{z + r}$ and substitute $\tilde{\theta}_\mathrm{s} = \xi^{-\gamma} f$. At $\xi \neq \sqrt{z_0 + r}$ the function f obeys the modified Bessel's equation

$$\mu\xi^2\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \mu\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} - \left(\gamma^2 + \frac{4s}{\mu}\xi^2\right) f = 0. \tag{42}$$

Therefore, two linearly independent solutions of Eq. 41 for $z \neq z_0$ can be chosen as

$$\tilde{\theta}_{s1}(z) = (z + r)^{-\gamma/2}\mathcal{I}_\gamma(2\sqrt{s(z + r)/\mu}), \tag{43}$$

$$\tilde{\theta}_{s2}(z) = (z + r)^{-\gamma/2}\mathcal{K}_\gamma(2\sqrt{s(z + r)/\mu}), \tag{44}$$

where $\mathcal{I}_\gamma$ and $\mathcal{K}_\gamma$ denote the modified Bessel functions of the first and second kind respectively Abramowitz and Stegun (1964).

Next it is straightforward to find that the function

$$\tilde{\theta}_s(z) = \begin{cases} A_1\tilde{\theta}_{s1}(z) + A_2\tilde{\theta}_{s2}(z), \; 0 \le z \le z_0, \\ \\ A_3\tilde{\theta}_{s2}(z), \qquad\qquad\qquad z \ge z_0, \end{cases} \tag{45}$$

with

$$A_1 = \frac{2N_0}{\mu}(z_0+r)^{\gamma/2}\mathcal{K}_\gamma\left(2\sqrt{\frac{s(z_0+r)}{\mu}}\right), \tag{46}$$

$$A_2 = \frac{\sqrt{\frac{sr}{\mu}}\mathcal{I}'_\gamma\left(2\sqrt{\frac{sr}{\mu}}\right) - \left(\frac{v_\mathrm{d}}{\mu}-\frac{\gamma}{2}\right)\mathcal{I}_\gamma\left(2\sqrt{\frac{sr}{\mu}}\right)}{\left(\frac{v_\mathrm{d}}{\mu}-\frac{\gamma}{2}\right)\mathcal{K}_\gamma\left(2\sqrt{\frac{sr}{\mu}}\right) - \sqrt{\frac{sr}{\mu}}\mathcal{K}'_\gamma\left(2\sqrt{\frac{sr}{\mu}}\right)}A_1, \tag{47}$$

$$A_3 = \frac{\mathcal{I}_\gamma\left(2\sqrt{\frac{s(z_0+r)}{\mu}}\right)}{\mathcal{K}_\gamma\left(2\sqrt{\frac{s(z_0+r)}{\mu}}\right)}A_1 + A_2, \tag{48}$$

satisfies Eq. 41 together with the boundary condition $[D(z)\partial_z\tilde{\theta}_\mathrm{s} + g\tau\tilde{\theta}_\mathrm{s}]_{z=0} = [v_\mathrm{d}\tilde{\theta}_\mathrm{s}]_{z=0}$.

The ground deposition flux $\tilde{j}_z(t) = -[v_\mathrm{d}\tilde{\theta}]_{z=0}$ is given by the following contour integral

$$\begin{aligned}\tilde{j}_z(t) &= -\frac{v_\mathrm{d}}{2\pi i}\int_C \mathrm{d}s\, e^{st}\left(A_1\tilde{\theta}_{s1}(0) + A_2\tilde{\theta}_{s2}(0)\right) \\ &= -\frac{N_0}{2\pi i}\frac{v_\mathrm{d}}{\mu}\int_C \mathrm{d}s\, e^{st}\left(1+\frac{z_0}{r}\right)^{\gamma/2}\frac{\mathcal{K}_\gamma(2\sqrt{s(z_0+r)/\mu})}{\left(\frac{v_\mathrm{d}}{\mu}-\frac{\gamma}{2}\right)\mathcal{K}_\gamma(2\sqrt{sr/\mu}) - \sqrt{\frac{sr}{\mu}}\mathcal{K}'_\gamma(2\sqrt{sr/\mu})}.\end{aligned} \tag{49}$$

Now we put $r \to 0$ and obtain

$$\tilde{j}_z(t) = -\frac{N_0}{i\pi\Gamma(\gamma)}\int_C \mathrm{d}s\, e^{st}\left(\frac{sz_0}{\mu}\right)^{\gamma/2}\mathcal{K}_\gamma\left(2\sqrt{\frac{sz_0}{\mu}}\right). \tag{50}$$

Note that the deposition velocity v_d drops out in this limit. The branch cut for the analytic continuation of the integrand in (50) is defined on the negative real axis of the complex plane. Then, we use so-called Hankel integration contour which extends from the point $-\infty - 0\cdot i$, around the origin counter-clockwise and back to the point $-\infty + 0\cdot i$. This leads us to the following result

$$\begin{aligned}\tilde{j}_z(t) = -\frac{N_0}{i\pi\Gamma(\gamma)}\left(\frac{z_0}{\mu}\right)^{\gamma/2}\int\limits_0^\infty s^{\gamma/2}e^{-st}\Big[e^{i\pi\gamma/2}\mathcal{K}_\gamma(2i\sqrt{sz_0/\mu}) \\ - e^{-i\pi\gamma/2}\mathcal{K}_\gamma(-2i\sqrt{sz_0/\mu})\Big]\mathrm{d}s = -\frac{N_0}{\Gamma(\gamma)}\frac{z_0^\gamma}{\mu^\gamma t^{\gamma+1}}\exp\left(-\frac{z_0}{\mu t}\right).\end{aligned} \tag{51}$$

The deposition flux $\tilde{j}_z$ and the number of airborne particles N are related by identity $\tilde{j}_z = \mathrm{d}N/\mathrm{d}t$. Performing integration of (51) over t we obtain the survival probability $p(t) \equiv N(t)/N_0$ in the explicit form (17).

Appendix 2

Here we derive the exact expression (27) for the surface distribution of deposited particles $\sigma(x)$. For this aim we turn to Eq. 3 and add the constant correction μr to diffusivity in order to avoid the vanishing of coefficient near highest spatial derivative. The resulting transport equation is as follows,

$$\partial_t\theta = \mu\partial_z[(z+r)\partial_z\theta]n + g\tau\partial_z\theta - \beta z^m\partial_x\theta. \tag{52}$$

The initial and boundary conditions are chosen to be $\theta(x,z,0) = N_0\delta(x)\delta(z-z_0)$ and $[\mu(z+r)\partial_z\theta + g\tau\theta]_{z=0} = [v_\mathrm{d}\theta]_{z=0}$.

The surface density $\sigma(x)$ is given by the total number of settled particles per unit length in the downwind direction, i.e.

$$\sigma(x) = -\int_0^{+\infty} j_z(x, z=0, t)\mathrm{d}t, \tag{53}$$

where $j_z = -\mu(z+r)\partial_z\theta - g\tau\theta$ is the vertical component of particle flux. Let us rewrite this relation as

$$\sigma(x) = -J_z(x, z=0), \tag{54}$$

where $J_z = -\mu(z+r)\partial_z\Theta - g\tau\Theta$ is the flux for integrated concentration $\Theta(x,z) = \int_0^\infty \theta(x,z,t)\mathrm{d}t$ which obeys the equation

$$\mu\partial_z[(z+r)\partial_z\Theta]n + g\tau\partial_z\Theta - \beta z^m\partial_x\Theta = -N_0\delta(x)\delta(z-z_0), \tag{55}$$

with the boundary condition $[\mu(z+r)\partial_z\Theta + g\tau\Theta]_{z=0} = [v_\mathrm{d}\Theta]_{z=0}$.

Next, we perform the Laplace transform of $\Theta(x,z)$ with respect to x

$$\Theta_\mathrm{s}(z) = \int_0^{+\infty} e^{-sx}\Theta(x,z)\mathrm{d}x. \tag{56}$$

This transformation leads to the following inhomogeneous differential equation for $\Theta_\mathrm{s}(z)$

$$\mu\partial_z[(z+r)\partial_z\Theta_\mathrm{s}]n + g\tau\partial_z\Theta_\mathrm{s} - s\beta z^m\Theta_\mathrm{s} = -N_0\delta(z-z_0). \tag{57}$$

For $0 \le z \le z_0$ the solution of this equation under the condition $[\mu(z+r)\partial_z\Theta + g\tau\Theta]_{z=0} = [v_\mathrm{d}\Theta]_{z=0}$ reads

$$\tilde{\theta}_s(z) = B_1\Theta_{s1}(z) + B_2\Theta_{s2}(z), \tag{58}$$

where

$$\Theta_{s1}(z) = (z+r)^{-\gamma/2}\mathcal{I}_{\gamma_m}\left(2\sqrt{\frac{\beta s}{\mu}}\frac{(z+r)^{\frac{m+1}{2}}}{m+1}\right), \tag{59}$$

$$\Theta_{s2}(z) = (z+r)^{-\gamma/2}\mathcal{K}_{\gamma_m}\left(2\sqrt{\frac{\beta s}{\mu}}\frac{(z+r)^{\frac{m+1}{2}}}{m+1}\right), \tag{60}$$

$$B_1 = \frac{2N_0}{(m+1)\mu}(z_0+r)^{\gamma/2}\mathcal{K}_{\gamma_m}\left(2\sqrt{\frac{\beta s}{\mu}}\frac{(z_0+r)^{\frac{m+1}{2}}}{m+1}\right), \tag{61}$$

$$B_2 = \frac{r^{\frac{m+1}{2}} \sqrt{\frac{\beta s}{\mu}} \mathcal{I}'_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{r^{\frac{m+1}{2}}}{m+1} \right) - \left(\frac{v_\mathrm{d}}{\mu} - \frac{\gamma}{2} \right) \mathcal{I}_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{r^{\frac{m+1}{2}}}{m+1} \right)}{\left(\frac{v_\mathrm{d}}{\mu} - \frac{\gamma}{2} \right) \mathcal{K}_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{r^{\frac{m+1}{2}}}{m+1} \right) - r^{\frac{m+1}{2}} \sqrt{\frac{\beta s}{\mu}} \mathcal{K}'_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{r^{\frac{m+1}{2}}}{m+1} \right)} B_1, \tag{62}$$

and $\gamma_m = \gamma/(m+1)$.

Now we apply the inverse Laplace transform to the solution (58) and substitute result into (54). This gives the surface density of particles in the form of an integral in the complex plane

$$\sigma(x) = \frac{v_\mathrm{d} N_0}{2\pi i} \int_C \frac{e^{sx} \left(1 + \frac{z_0}{r}\right)^{\gamma/2} \mathcal{K}_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{(z_0+r)^{\frac{m+1}{2}}}{m+1} \right)}{\left(v_\mathrm{d} - \frac{\mu\gamma}{2}\right) \mathcal{K}_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{r^{\frac{m+1}{2}}}{m+1} \right) - r^{\frac{m+1}{2}} \sqrt{\mu\beta s} \mathcal{K}'_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{r^{\frac{m+1}{2}}}{m+1} \right)} \mathrm{d}s, \tag{63}$$

which in the limit $r \to 0$ becomes

$$\sigma(x) = \frac{N_0 z_0^{\gamma/2} \beta^{\gamma_m/2}}{\mu^{\gamma_m/2} (m+1)^{\gamma_m} \Gamma(\gamma_m) \pi i} \int_C e^{sx} s^{\gamma_m/2} \mathcal{K}_{\gamma_m} \left(2\sqrt{\frac{\beta s}{\mu}} \frac{z_0^{\frac{m+1}{2}}}{m+1} \right) \mathrm{d}s. \tag{64}$$

To calculate this integral we define contour C as the Hankel path extending from the point $-\infty - 0 \cdot i$, circling the origin counter-clockwise, and returning to the point $-\infty + 0 \cdot i$. After some algebra the closed form expression (27) can be derived.

Appendix 3

Here we consider the case where the surface source is switched on at $t = 0$. For definiteness, we speak about the dust produced by the industrial area. Evolution of the dust concentration is described by (39). In this sub-section we put $\mu \to 1$ thus passing to

$$\partial_t \tilde{\theta} = \partial_z \left\{ [(z+r)\partial_z + \gamma] \tilde{\theta} \right\}. \tag{65}$$

with the initial condition $\tilde{\theta}(z, 0) = 0$ We consider two different boundary conditions corresponding to a fixed dust concentration at $z = 0$ and to a fixed particle flux at $z = 0$.

Let us produce the Laplace transform with respect to time

$$\tilde{\theta}_\mathrm{s}(z) = \int_0^\infty \mathrm{d}t \, \exp(-st) \tilde{\theta}(z, t). \tag{66}$$

The inverse Laplace transform reads

$$\tilde{\theta}(z, t) = \int_C \frac{\mathrm{d}s}{2\pi i} \exp(st) \tilde{\theta}_\mathrm{s}(z), \tag{67}$$

where integration is performed along a line parallel to the imaginary axis to the right of all singularities of $\tilde{\theta}_\mathrm{s}$. In terms of $\tilde{\theta}_\mathrm{s}$ the Eq. 65 becomes

$$s\tilde{\theta}_\mathrm{s} = \partial_z \left\{ [(z+r)\partial_z + \gamma] \tilde{\theta}_\mathrm{s} \right\}. \tag{68}$$

A solution of the Eq. 68 tending to zero as $z \to \infty$ is

$$\tilde{\theta}_\mathrm{s}(z) \propto (z+r)^{-\gamma/2} K_\gamma (2\sqrt{s(z+r)}). \tag{69}$$

Let us pose the boundary condition $\tilde{\theta} = 1$ at $z = 0$. In terms of the Laplace transform the boundary condition is $\tilde{\theta}_s(0) = s^{-1}$. The solution (69) with the boundary condition is

$$\tilde{\theta}_s(z) = \frac{1}{s}\left(\frac{r}{z+r}\right)^{\gamma/2} \frac{K_\gamma(2\sqrt{s(z+r)})}{K_\gamma(2\sqrt{sr})}. \tag{70}$$

Now we should perform the inverse Laplace transform. When (70) is substituted into Eq. 67 one finds

$$\tilde{\theta}(z,t) = \int_C \frac{ds}{2\pi i s}\left(\frac{r}{z+r}\right)^{\gamma/2} \frac{K_\gamma(2\sqrt{s(z+r)})}{K_\gamma(2\sqrt{sr})}\exp(st). \tag{71}$$

For small values of argument $K_\gamma(x) = 2^{-1}\Gamma(\gamma)(2/x)^\gamma$. Substituting the expression into Eq. 71 one obtains

$$\begin{aligned}\tilde{\theta}(z,t) &= \frac{r^\gamma}{\Gamma(\gamma)}\int_C \frac{ds}{\pi i s}\left(\frac{s}{z+r}\right)^{\gamma/2} K_\gamma(2\sqrt{s(z+r)})\exp(st)\\ &= \frac{1}{\Gamma(\gamma)}\frac{r^\gamma}{(z+r)^\gamma}\left(\frac{z+r}{t}\right)^{\gamma/2}\int_C \frac{d\zeta}{\pi i \zeta}\zeta^{\gamma/2}K_\gamma(2\sqrt{\zeta(z+r)/t})\exp(\zeta).\end{aligned} \tag{72}$$

If $z \ll t$ then the main contribution to the integral (72) is produced by the residue in the pole $\zeta = 0$, therefore

$$\tilde{\theta} \approx (r/z)^\gamma, \tag{73}$$

for $z \gg r$. If $z \gg t$ then the main contribution to the integral (72) stems from the saddle point $\zeta = (z+r)/t$, therefore

$$\tilde{\theta} \propto \left(\frac{h}{z}\right)^\gamma \left(\frac{z}{t}\right)^{\gamma-1}\exp\left(-\frac{z}{t}\right), \tag{74}$$

for $z \gg r$.

Let us now consider another boundary condition at $z = 0$ that implies constant upward flux of particles:

$$-r\partial_z\tilde{\theta} - \gamma\tilde{\theta} = 1. \tag{75}$$

We focus on the the case $\gamma < 1$. Then the main terms of the McDonald function expansion at small x are

$$K_\gamma(x) \approx \frac{1}{2}\left[\Gamma(\gamma)\left(\frac{x}{2}\right)^{-\gamma} + \Gamma(-\gamma)\left(\frac{x}{2}\right)^{\gamma}\right].$$

Therefore the boundary condition (75) leads to

$$\tilde{\theta}_s(z) = \frac{1}{s^{1+\gamma/2}(z+r)^{\gamma/2}}\frac{K_\gamma(2\sqrt{s(z+r)})}{\Gamma(1-\gamma)}. \tag{76}$$

Substituting the expression into Eq. 67 we find

$$\begin{aligned}\tilde{\theta}(z,t) &= \int_C \frac{ds}{2\pi i}\exp(st)\frac{1}{s^{1+\gamma/2}(z+r)^{\gamma/2}}\frac{K_\gamma(2\sqrt{s(z+r)})}{\Gamma(1-\gamma)}\\ &= \left(\frac{t}{z+r}\right)^{\gamma/2}\int_C \frac{d\zeta}{2\pi i \zeta^{1+\gamma/2}}\exp(\zeta)\frac{K_\gamma(2\sqrt{\zeta(z+r)/t})}{\Gamma(1-\gamma)}.\end{aligned} \tag{77}$$

At small z/t the integral is gained at $\zeta \sim 1$. Substituting the main asymptotic of the McDonald function and deforming the integration contour to the negative semi-axis, one obtains

$$\tilde{\theta}(z,t) = \frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}\left(\frac{t}{z}\right)^{\gamma}\int_C \frac{d\zeta}{4\pi i\zeta^{1+\gamma}}\exp(\zeta) = \frac{\Gamma(\gamma)}{2\Gamma(1+\gamma)\Gamma(1-\gamma)}\left(\frac{t}{z}\right)^{\gamma}, \tag{78}$$

for $z \gg r$. At large z/t the integral is determined by the saddle point where the asymptotic expression for K_γ can be exploited. As a result, one finds

$$\tilde{\theta} \propto \left(\frac{t}{z}\right)^{1+\gamma} \exp\left(-\frac{z}{t}\right), \tag{79}$$

for $z \gg r$. The expressions (78,79) demonstrate the same self-similarity as the expressions (73,74) do, however, with other power prefactor.

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