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Vorticity statistics in the direct cascade of two-dimensional turbulence

Gregory Falkovich[1] and Vladimir Lebedev[2]

[1]*Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel*

[2]*Landau Institute for Theoretical Physics and Moscow Institute of Physics and Technology, 119334 Moscow, Russia*

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For the direct cascade of steady two-dimensional (2D) Navier-Stokes turbulence, we derive analytically the probability of strong vorticity fluctuations. When ϖ is the vorticity coarse-grained over a scale R, the probability density function (PDF), $\mathcal{P}(\varpi)$, has a universal asymptotic behavior $\ln \mathcal{P} \sim -\varpi/\varpi_{\rm rms}$ at $\varpi \gg \varpi_{\rm rms} = [H \ln(L/R)]^{1/3}$, where H is the enstrophy flux and L is the pumping length. Therefore, the PDF has exponential tails and is self-similar, that is, it can be presented as a function of a single argument, $\varpi/\varpi_{\rm rms}$, in distinction from other known direct cascades.

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As "every unhappy family is unhappy in its own way," nonequilibrium systems are expected to differ by the way they deviate from equilibrium. All the more remarkable are thus any universal features one can distinguish in the classes of nonequilibrium systems. Turbulence is a paradigmatic far-from-equilibrium state and the central question of physics of turbulence is also that of universality: how much one needs to know about external forcing to predict flow statistics. A related question is that of symmetries of the statistics, particularly whether scale invariance appears for the scales distant from L, where turbulence is produced [1]. One distinguishes direct and inverse cascades occurring at the scales much smaller or much larger than L, respectively. Data suggest that the statistics of inverse cascades are scale invariant [1–6] with some aspects being even conformal invariant [7,8]. For example, the PDF of ϖ, that is, the vorticity $\omega = \nabla \times \mathbf{v}$ spatially averaged over the disk of radius R, is empirically found to be a function of a single variable rather than two in two-dimensional (2D) inverse cascade: $\mathcal{P}(\varpi, R) = \varpi^{-1} f(\varpi R^{-2/3})$ [3–6]. Such self-similarity was never observed in direct cascades; $\mathcal{P}(\varpi, R)$ changes the form as the ratio R/L varies [1,2].

One way to explain this profound difference between direct and inverse cascades is to argue that fluid motions are slower when scales are larger. As an inverse cascade proceeds upscale, it has ample time to be effectively averaged over small-scale fluctuations including those of the pumping, whose only memory left is the flux value it generates. On the contrary, small-scale fast fluctuations in a direct cascade stay sensitive to the statistics of slow fluctuations at large scales [9]; nonlinearity enhances variability down the cascade so that small-scale statistics is dominated by rare strong fluctuations. One can also explain the difference between direct and inverse cascades using the Lagrangian language. Correlation functions are accumulated along the Lagrangian trajectories. Inverse cascades are related to trajectories approaching each other back in time, then two-particle behavior effectively determines the evolution of multiparticle configurations and the second moment determines the scaling of higher moments. On the contrary, direct cascades correspond to trajectories separating back in time, one then relates the breakdown of scale invariance at vanishing viscosity to nonuniqueness of explosively separating trajectories in a nonsmooth velocity field; exponents of higher moments are then related to the laws of decay of the fluctuations of the shapes of multiparticle configurations. These laws depend on the number of particles so that an infinite number of forcing-related parameters is needed to predict the statistics at small scales [2].

Prior knowledge was based on experimental and numerical data; the only analytical results were obtained for passive fields in synthetic flows [2]. Here, for the first time, the vorticity PDF tail is analytically derived from the equation of motion. We consider the direct (enstrophy) cascade of 2D turbulence [10–12], whose physical mechanism is that pumping-produced vorticity blobs are deformed by the flow into thin streaks until viscosity dissipates them. In Lagrangian terms, such turbulence is peculiar since it corresponds to an exponential separation of trajectories. Indeed, constancy of the enstrophy flux over scales, $H = \nabla \langle (\boldsymbol{v}_1 - \boldsymbol{v}_2)\omega_1\omega_2 \rangle =$ const, suggests the scaling $|\boldsymbol{v}_1 - \boldsymbol{v}_2| \propto |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ (i.e., spatially smooth velocity). In a steady state, the enstrophy dissipation $\nu |\nabla \omega|^2$ must stay finite in the inviscid limit $\nu \to 0$. The velocity then cannot be perfectly smooth, but the possible singularities are no stronger than logarithmic [13,14]. If one assumes self-similarity in a sense that $\mathcal{P}(\varpi, R) = \varpi^{-1} f[\varpi^a / \ln(L/R)]$, then the flux constancy requires $a = 3$ [10,13,14]. Further using the self-similarity assumption, one estimates the enstrophy transfer time through a given scale R, determined by the stretching and contraction rate, as a turnover time or an inverse vorticity at this scale. On the one hand, that time decreases with the scale as $\ln^{-1/3}(L/R)$, which would suggest that the small-scale statistics is sensitive to the statistics at larger scales. On the other hand, the total time of enstrophy transfer from L down to the viscous scale η diverges $\propto \ln^{2/3}(L/\eta)$ as $\eta \to 0$. Particle trajectories are then expected to separate exponentially rather than explosively and stay unique even in the inviscid limit, that makes self-similarity plausible, according to the above Lagrangian arguments.

Von Neumann [15] and Kraichnan [10] argued that an infinite number of vorticity conservation laws can make the vorticity cascade nonuniversal; we later countered that the fluxes of higher vorticity invariants must be irrelevant due to the phenomenon of "distributed pumping" [13]. Self-similarity breakdown was found empirically for the vorticity isolines, which are conformal invariant in the inverse cascade, while in the direct cascade they are not scale invariant but multifractal with the fractal dimension 3/2 and higher dimensions saturating at 1 [7,8] (that may be related to strain persistence that leads to long thin streaks of vorticity); that leads one to expect that the bulk vorticity statistics is not self-similar either.

Here we analytically derive the (non-Gaussian) tail of the PDF of the vorticity ϖ coarse-grained over the scale R in the direct (enstrophy) cascade of 2D turbulence permanently pumped by an external force. We show that the tail is exponential,

$$\ln \mathcal{P}(\varpi,R) \sim -|\varpi|[H\ln(L/R)]^{-1/3}, \tag{1}$$

for a driving force with a finite correlation time. In particular, Eq. (1) shows that the PDF is self-similar, that is, it can be presented as $\mathcal{P}(\varpi,R)=\varpi^{-1}f[\varpi^3/\ln(L/R)]$. To obtain the tail of the single-point vorticity PDF, the ratio $L{:}R$ should be substituted by $L/\eta=\sqrt{Re}$ in Eq. (1).

We start with the forced 2D Navier-Stokes equation:

$$\partial\omega/\partial t+(\boldsymbol{v}\nabla)\,\omega=\nu\nabla^2\omega+\phi, \tag{2}$$

where $\nabla\cdot\boldsymbol{v}=0$ due to incompressibility. The pumping ϕ is assumed to be a random Gaussian field spatially correlated on the scale L and short correlated in time: $\langle\phi(0,\mathbf{0})\phi(t,\boldsymbol{r})\rangle=\delta(t)\chi(r)$, where $\chi(r)\to 0$ as $r/L\to\infty$ and $\chi(0)=H$. We show below that the processes that contribute to the vorticity PDF tails take a long time which allows effective averaging over forcing so that our results are asymptotically valid for any force with a finite correlation time. The viscous term will be ignored as long as we consider flow fluctuations on scales larger than η.

The vorticity statistics can be described by the Martin-Siggia-Rose formalism [16] where averages are path integrals, $\int \mathcal{D}p\mathcal{D}\omega\exp(i\mathcal{I})\ldots$, with the effective action

$$\mathcal{I}=\int dt d^2r p\left[\partial_t\omega+\boldsymbol{v}\nabla\omega+\frac{i}{2}\int d^2r_1\chi(\boldsymbol{r}-\boldsymbol{r}_1)p(\boldsymbol{r}_1)\right].$$

The field p is introduced to put (2) into the exponent and to average then over the ϕ statistics. Since the action contains a cubic term originating from the nonlinear term in Eq. (2), one is unable to calculate the path integrals explicitly, nor use perturbation theory since the coupling is strong. We examine tails of $\mathcal{P}(\varpi,r)$ using $\varpi/\varpi_{\rm rms}$ as a large parameter and calculating the path integral in the saddle-point approximation, employing the so-called instanton formalism adapted for turbulence in [17]. In this way, one looks for an action extremum, defined by the instanton equations $\delta\mathcal{I}/\delta\omega=0=\delta\mathcal{I}/\delta p$ with appropriate boundary conditions. Both the action and the measured quantity ϖ are invariant with respect to rotations and so are instanton equations and their boundary conditions. However, axial symmetry turns nonlinearity in the instanton equations into zero (i.e., a "naive instanton" is meaningless). The physical reason is quite transparent: There is neither stretching nor contraction for axially symmetric flows so that the force can pump the vorticity forever. The flow realizations that determine ϖ must have their axial symmetry broken. We establish below that the angle-dependent part of the vorticity remains much smaller than the isotropic part during most of the evolution (by virtue of the large parameter $\varpi/\varpi_{\rm rms}$). That will allow us to integrate over the angle-dependent degrees of freedom (in the Gaussian approximation) and obtain a renormalized action for the zero angular harmonic ω_0. Moreover, we show that only the second angular harmonic provides for the relevant renormalization by virtue of the large parameter $\ln(L/R)$. We then find the new (effectively axially symmetric) instanton that corresponds to the renormalized action and gives the tails of $\mathcal{P}(\varpi,R)$.

To realize this program for the enstrophy cascade, we use a Lagrangian frame attached to a fluid particle. In polar coordinates, $\boldsymbol{r}=(r\cos\varphi,r\sin\varphi)$, we expand ω and p over the angular harmonics $\omega(\boldsymbol{r})=\sum\omega_m(r)\exp(im\varphi)$, $2\pi p(\boldsymbol{r})=\sum p_m(r)\exp(im\varphi)$. As shown below, slow strongly fluctuating degrees of freedom are described by ω_0, p_0. It makes no sense to simply omit the degrees of freedom related to high angular harmonic in the action $\mathcal{I}$, since the resulting action $\mathcal{I}_0$ does not describe any deformation of ω_0. Therefore, one has to account for ω_m, p_m to obtain an effective action $\mathcal{I}_{\rm eff}$ for the zero harmonics:

$$\exp(i\mathcal{I}_{\rm eff})=\int\prod_{m>0}\mathcal{D}\omega_{\pm m}\,\mathcal{D}p_{\pm m}\,\exp\left(i\mathcal{I}\right). \tag{3}$$

The action $\mathcal{I}_{\rm eff}$ will then describe deformations of the zero angular harmonic ω_0 induced by fluctuations of high harmonics. The action has a number of terms, $\mathcal{I}=\mathcal{I}_0+\mathcal{I}_3+\sum_m(\mathcal{I}_m+\mathcal{I}_{-m}+\mathcal{I}_{im})$. Here $\mathcal{I}_0$ contains only p_0,ω_0. The quadratic terms are $\mathcal{I}_{im}=-\int dt\,d^2r\partial_r p_0(v_m\omega_{-m}+v_{-m}\omega_m)$ and $\mathcal{I}_m$ containing p_mp_{-m}, $p_{-m}\omega_m$. The term $\mathcal{I}_3$ is cubic in p_m,ω_m with $m\neq 0$; it is small and neglected in what follows. That allows one to derive an effective action for the zero harmonic, $\mathcal{I}_{\rm eff}=\mathcal{I}_0+\Delta\mathcal{I}$, by integrating over other harmonics in the Gaussian approximation. If $\mathcal{I}_{im}=0$ then $\Delta\mathcal{I}=0$, since any path integral of the form $\int\mathcal{D}p\mathcal{D}\omega\,p(\omega_t+\ldots)$ is unity due to causality. Therefore,

$$\Delta\mathcal{I}=\sum_m\sum_{n=1}^{\infty}\frac{i^{n-1}}{n!}\langle\langle(\mathcal{I}_{im})^n\rangle\rangle, \tag{4}$$

where the double brackets denote cumulants obtained by integration over $\omega_{\pm m},p_{\pm m}$ with the weight $\exp(i\mathcal{I}_m+i\mathcal{I}_{-m})$. Consistently considering small fluctuations (as in neglecting $\mathcal{I}_3$) we take only the term with $n=1$, which is determined by the pair correlation functions, $F_m=\langle\omega_m(t,r_1)\omega_{-m}(t,r_2)\rangle$.

We pass to the logarithmic variable $\xi=\ln(r/L)$ and consider small scales: $r\ll L$. The term $\langle\mathcal{I}_{i2}\rangle$ in Eq. (4) contains an extra power of $|\xi|\gg 1$ (as noticed already in [13] and is likely related to peculiarity of elliptic vortices) so we retain only this term:

$$\Delta\mathcal{I}\approx i\int dt\,d\xi\,q(t,\xi)\int_\xi d\zeta\,[F_2(\zeta,\xi)-F_2(\xi,\zeta)], \tag{5}$$

where $q(\xi)=r^2p_0(r)$. Here F_2 is a functional of ω_0 to be extracted from the equation that in the main logarithmic approximation is written as follows [18]:

$$\partial_t F_2(\xi_1,\xi_2)+i\,\hat{O}_1F_2-i\,\hat{O}_2F_2=\chi_2\,,$$
$$\hat{O}f(\xi)=\int d\xi'\left[\omega_0\delta(\xi'-\xi)+\frac{\theta(\xi'-\xi)}{2}\partial_\xi\omega_0(\xi)\right]f(\xi')\,. \tag{6}$$

Here $\chi_2(r_1,r_2)=\int d\varphi\,\cos(2\varphi)\chi\,(\boldsymbol{r}_1-\boldsymbol{r}_2)/2\pi$ and θ is the step function. One expresses (5) via (6) at $\xi_1=\xi_2$:

$$\Delta\mathcal{I}=-2\int dt\,d\xi\,q(t,\xi)[\partial_\xi\omega_0(t,\xi)]^{-1}\partial_tF_2(t,\xi,\xi)\,. \tag{7}$$

This term describes how fluctuations distorted by a strong vortex ω_0 act back on the vortex.

To solve (6), we find the right and left eigenfunctions of the operator $\hat{O}$, respectively,

$$\varphi_\lambda = \theta(\omega_0 - \lambda) 2\partial_\xi \sqrt{\omega_0 - \lambda}\,,$$
$$\phi_\mu(\xi) = (2\pi)^{-1} \lim_{\epsilon\to 0} \mathrm{Re}\,[\mu - \omega_0(\xi) + i\epsilon]^{-3/2}\,,$$

and expand $F_2(\xi,\zeta) = \int d\lambda\, d\mu\, \Phi(\lambda,\mu)\varphi_\lambda(\xi)\varphi_\mu(\zeta)$ in (6):

$$[\partial_t + i(\mu-\nu)]\Phi(t,\mu,\nu) + \int d\lambda\, \Phi(\lambda,\nu) J(\mu,\lambda) + \int d\lambda \Phi(\mu,\lambda) J(\nu,\lambda) = \int d\xi\, d\eta\, \phi_\mu(\xi)\phi_\nu(\eta)\chi_2(\xi,\eta), \quad (8)$$

$$J(\mu,\lambda) \equiv \int d\zeta\, \phi_\mu(\zeta)\partial_t \varphi_\lambda(\zeta) \approx \delta'(\mu-\lambda)\psi(\lambda) + \delta(\mu-\lambda)\psi'(\lambda)/2, \quad (9)$$

where $\psi(\omega_0) = \partial_t \omega_0$. In the last line we used the adiabatic approximation, since the instanton is shown below to change slowly on its own rotation timescale ω_0^{-1}. Substituting (9) into (8) we get the first-order equation, which we solve by the method of characteristics. The initial condition for this equation is posed at some distant past moment t_* where ω_0 is of the order of a typical (rms) fluctuation and is some slow (logarithmic) function of the distances in the region $|\xi| \lesssim \ln(L/R)$ so that $\partial_\xi \ln \omega_0 \sim \xi^{-1}$. We assume that at $t = t_*$ the second moment is $\langle \omega(\boldsymbol{r}_1)\omega(\boldsymbol{r}_2)\rangle \simeq H^{2/3} \ln^{2/3}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|/L)$. Strictly speaking, we cannot derive that from the equation of motion. That choice is consistent with the flux relation and, as we show below, is self-consistent with the higher moments described by the PDF tail to be derived. The second angular harmonic is then $F_2(t_*,\xi_1,\xi_2) = \int_0^{2\pi} e^{-2i\varphi} \langle \omega(\boldsymbol{r}_1)\omega(\boldsymbol{r}_2)\rangle (d\varphi/2\pi) \approx H^{2/3}|\xi_1|^{-1/3}\delta(\xi_1 - \xi_2)$ and $\Phi(t_*,\lambda,\mu) \sim (H\xi)^{2/3}(\lambda\mu)^{-1/2}\delta'(\lambda-\mu)$. With that we obtain the homogeneous solution,

$$\frac{F_2(t,\xi,\xi)}{t-t_*} \sim H^{2/3}(\partial_\xi\omega_0)^2 \int d\zeta\, \zeta^{2/3}\partial_\zeta \ln\omega_0(t_*,\zeta) \times \frac{\theta[\omega_0(t,\xi)-\omega_0(t,\zeta)]}{\omega_0(t,\xi)-\omega_0(t,\zeta)} \sim \frac{H^{2/3}\partial_\xi\omega_0}{\xi^{1/3}}\mathcal{L}, \quad (10)$$

where $\mathcal{L} = \ln|\omega_0/\partial_\xi\omega_0| \simeq \ln\ln(L/R)$. At the derivation we used $\partial_\zeta \ln\omega_0(t_*,\zeta) \simeq 1/\zeta$ and cut off the logarithmic divergence due to a finite (order-unity) width of $F_2(t_*,\xi_1,\xi_2)$ over $\xi_1 - \xi_2$. In much the same way one can obtain the inhomogeneous (pumping-generated) solution: $F_2^{\mathrm{pump}}(\xi,\xi) \sim H^{1/3}\omega_0{}^{-1}(\partial_\xi\omega_0)^2$, which is much smaller since we shall obtain a slow instanton with $H^{1/3}t_* \gg 1$. That means that the pumping-produced anisotropic fluctuations give a smaller contribution than deformation of an initial fluctuation. The consequence is that the tail of the vorticity PDF is insensitive to the form of the pumping correlation function and is determined solely by its zeroth moment, that is, the vorticity flux H. That means universality of the statistics of strong vorticity fluctuations.

Substituting (10) into (7) one obtains $\Delta\mathcal{I}$ and then the effective action,

$$\mathcal{I}_{\mathrm{eff}} = \Delta\mathcal{I} + \int dt\, d\xi\, q\,\partial_t\omega_0 + \frac{iH}{2}\int dt\, d\xi_1\, d\xi_2\, q(\xi_1)q(\xi_2).$$

After some rescaling of the fields the action gives the following instanton equations:

$$\partial_t\omega_0 = (H^{2/3}\xi^{-1/3}/\partial_\xi\omega_0)\partial_t[(t-t_*)\partial_\xi\omega_0\mathcal{L}] + HQ, \quad (11)$$
$$\partial_t q = -\partial_\xi\{\mathcal{L}(H^{2/3}\xi^{-1/3}/\partial_\xi\omega_0)\partial_t[(t-t_*)q]\}, \quad (12)$$

where $Q = -i\int d\zeta\, q(\zeta,t)$. In deriving (12) we exploited the large value of the logarithm $\ln(L/R)$ treating $\mathcal{L}$ as a constant. Apart from the logarithm $\mathcal{L}$, the correction to the action $\Delta\mathcal{I}$ depends only on the vorticity spatial derivative $\partial_\xi\omega_0$. As a result, the variation with respect to ω_0 gives (12), which is a continuity equation, so that $dQ/dt = 0$ in the main order in the large logarithm. The first term in the rhs of (11) is negative at $\xi < 0$ that is the correction (7) describes decrease of the vorticity due to deformation of the circular vortex by elliptic perturbations. Since Q is t independent, then ω_0 grows linearly: $\omega_0 = [2H^{2/3}\xi^{-1/3}\ln|\xi| + HQ](t-t_*)$. With the logarithmic accuracy, $\varpi = \omega_0[0,\ln(L/R)]$, that enables one to express t_* via ϖ and Q. Then one substitutes it into the total action, optimizes over Q and finds

$$\ln\mathcal{P} \approx i\mathcal{I}_{\mathrm{eff}}^{\mathrm{saddle}} \approx -H\int dt\, Q^2/2 \approx -HQ^2|t_*|/2 \simeq -4H^{-1/3}\varpi[\ln(L/R)]^{-1/3}\ln[\ln(L/R)]. \quad (13)$$

Omitting here the slow factor $\ln[\ln(L/R)]$ one obtains Eq. (1). The value of $\omega_0(t_*)$ does not enter as long as $\omega_0(t_*) \ll \varpi$.

The instanton solution thus found enables one to check the assumptions made in deriving $\mathcal{I}_{\mathrm{eff}}$. The applicability condition of the saddle-point approximation is $|\varpi^3| \gg H\ln(L/R)$ and we consider the inertial interval where $\ln(L/R) \gg 1$. The fluctuations on the background of our instanton are indeed small: $F_2 \simeq t_*\omega_0\xi^{-4/3}\omega_0^2/\xi \ll \omega_0^2$ as was assumed. Higher correlation functions are smaller than F_2, which justifies neglecting $\mathcal{I}_3$ and $n > 1$ terms in (4). The instanton lifetime $|t_*| \gg 1/\varpi$, that is indeed $\omega_0(t)$ changes weakly during the time ω_0^{-1}.

It is illuminating to compare the vorticity statistics in the direct 2D cascade with the statistics of the passive scalar in a spatially smooth random flow. For the scalar ϑ coarse-grained over a scale R less than the pumping length L, the asymptotic behavior of the single-point PDF in a smooth random flow is given by the following reasoning. Large values of ϑ are achieved when there is no stretching for a time much longer than the mean stretching time $\lambda^{-1}\ln(L/R)$, where λ is the Lyapunov exponent. During that time, the passive scalar is pumped by a random forcing (i.e., it has Gaussian statistics with the linearly growing variance):

$$\mathcal{P}(\vartheta) \sim \int dt\, \mathcal{Q}(t)\exp(-\vartheta^2/Pt), \quad (14)$$

where $\mathcal{Q}(t)$ is the probability of no stretching during time t. Stretching is correlated on the velocity timescale λ^{-1}, which is independent of ϑ. For every stretching event, the scalar blob is stretched by order e and we ask for the probability that there were less than the number $\ln Pe$ such events during t.

For $t \gg \lambda_0^{-1}$, this is the probability of the Poisson process: $\ln \mathcal{Q}(t) \sim -\lambda t + O[\ln(L/r)]$. Saddle-point integration over t in (14) gives the exponential tail (first suggested in [19,20] and derived by the instanton formalism in [17,21]): $\ln[\mathcal{P}(\vartheta)] \sim -\vartheta\sqrt{\lambda/P} + O[\ln(L/R)]$. For the vorticity cascade, we use similar reasoning with the knowledge added from [13] that the stretching correlation time is the mean total stretching time from R to L, which is $H^{-1/3}\ln^{2/3}(L/R)$. That gives

$$\mathcal{P}(\varpi) \sim \int dt\ \exp[-\varpi^2/Ht - tH^{1/3}\ln^{-2/3}(L/R)].$$

Saddle-point $t \sim \varpi \ln^{1/3}(L/R)H^{-2/3}$ coincides with t_* from the instanton solution, and the integration reproduces (1). We see that vorticity is indeed like passive scalar: The stronger the fluctuation the longer it lives, which gives sub-Gaussian PDF tails.

Our analytic result (1) is supported by two different sets of simulations, which both show that the vorticity PDF tails are approximately exponential [22,23]. In addition, Fig. 5 from [23] supports our conclusion that the PDF is getting self-similar in the inertial interval. Data with different pumping statistics and better resolution are needed for a quantitative comparison. The broader significance of our work is that it shows one how to describe the statistics of strong fluctuations that take a long time to build up: obtain an effective action for the slowest variable by integrating over faster degrees of freedom, then apply the saddle-point approximation. Such an approach is unlikely to be applicable to direct cascades with power-law temporal acceleration (like three-dimensional energy cascade) where strong fluctuations are expected to be fast not only for vorticity but even for velocity [24]. It may work, however, for decelerating cascades, both direct and inverse. Plenty of such cascades can be found in wave turbulence [25]; another task is to apply this approach to the PDF tail of ϖ in the inverse cascade of 2D turbulence, where one may expect relevance of slow events.

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