Structure of coherent vortices generated by the inverse cascade of two-dimensional turbulence in a finite box

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We discuss the structure and geometrical characteristics of coherent vortices appearing as a result of the inverse cascade in two-dimensional turbulence in a finite box. We demonstrate that the universal velocity profile, established by J. Laurie *et al.* [Phys. Rev. Lett. **113**, 254503 (2014)], corresponds to the passive regime of flow fluctuations. We find the vortex core radius and the vortex size, and we argue that the amount of vortices generated in the box depends on the system parameters.

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I. INTRODUCTION

The role of the counteraction of turbulence fluctuations with a mean (coherent) flow is one of the central problems of turbulence theory [1]. Usually the fluid energy is transferred from the large-scale flow to turbulent pulsations [2]. However, in some cases the energy can go from small-scale fluctuations to large-scale ones, which can lead to the formation of a mean flow [3]. Basic problems, such as how to determine the mean velocity at which turbulent fluctuations are sustained, remain the object of intense investigations [4]. There is still no consistent theory for the mean (coherent) profile coexisting with turbulent fluctuations, thus even the celebrated logarithmic law for the turbulent boundary layer is a subject of controversy [5]. Here, we consider two-dimensional (2D) turbulence in a restricted box where large-scale coherent structures are generated from small-scale fluctuations excited by pumping. This process occurs because in two dimensions, the nonlinear interaction favors the energy transfer to larger scales [6–8].

Initial experiments on 2D turbulence [9] have already shown that in a finite system with small bottom friction, the energy transfer to large scales leads to the formation of coherent vortices. Numerical simulations [10–12] also demonstrate the appearance of coherent vortices in 2D turbulence. Subsequent numerical simulations [13] and experiments [14] demonstrated that these vortices have well-defined and reproducible mean velocity (vorticity) profiles. This profile is quite isotropic with a power-law radial decay of vorticity inside the vortex. The profile in that region depends neither on the boundary conditions (no-slip in experiments, periodic in numerics) nor on the type of forcing (random in numerics versus parametric excitation or electromagnetic force in experiments). The same profile is formed both in the statistically stationary case where the mean flow level is stabilized by the bottom friction, and in the case in which the average flow is still not stabilized and increases as time passes.

In Ref. [15], the results of intensive simulations of 2D turbulence were reported, and they demonstrated that the vortex polar velocity profile is flat in some interval of distances from the vortex center. That means that the average vorticity is inversely proportional to the distance r from the vortex center. In the same paper, a theoretical scheme based on conservation laws and symmetry arguments was proposed that explains the flat velocity profile. The scheme predicts the value of the polar velocity $U=\sqrt{3\epsilon/\alpha}$ (where ϵ is the energy production rate and α is the bottom friction coefficient), which is in excellent agreement with the numerics [15]. However, in the numerics the region of existence of the flat profile is definitely restricted. In addition, in early simulations [10–12] no flat velocity profile was observed. These facts require an explanation.

We performed a detailed analytical investigation of the problem of 2D turbulence in a finite box. As a result, we established that the flat velocity profile corresponds to the passive regime of the flow fluctuations where their self-interaction can be neglected. The passive regime allows for consistent analytical calculations that confirm the validity of the value $U=\sqrt{3\epsilon/\alpha}$ for the polar velocity. In addition, we found expressions for the viscous core radius of the vortex and for the border of the region where the flat velocity profile is realized. The results explain why no flat velocity profile was observed in early simulations [10–12], and they imply that at some conditions a large number of coherent vortices could appear instead of a few vortices in numerics [13,15] and experiment [14].

II. GENERAL RELATIONS

We consider the case in which 2D turbulence is excited in a finite box of size L by external forcing. It is assumed to be random with statistical properties that are homogeneous in time and space. We assume also that the correlation functions of the force are isotropic. The main object of our investigation is the stationary (in the statistical sense) turbulent state that is caused by such forcing.

For exciting turbulence, the forcing should be stronger than dissipation that is caused by both bottom friction and viscosity. That implies that the characteristic velocity gradient of the fluctuations produced by the forcing should be much larger than their damping at the pumping scale. The velocity gradient is estimated as $\epsilon^{1/3}k_f^{2/3}$, where ϵ is the energy production rate per unit mass and k_f is the characteristic wave vector of the pumping force. Thus we arrive at the inequalities

$$\epsilon^{1/3}k_f^{2/3} \gg \alpha,\gamma, \tag{1}$$

where α is the bottom friction coefficient, and γ is the viscous damping rate at the pumping scale k_f^{-1}, $\gamma=\nu k_f^2$ (ν is the kinematic viscosity coefficient). In simulations, hyperviscosity is often used. In that case, the inequalities (1) are still obligatory for exciting turbulence, where γ is the hyperviscous damping rate at the pumping scale k_f^{-1}.

If the inequalities (1) are satisfied, then turbulence is excited and pulsations of different scales are formed due to nonlinear interaction of the flow fluctuations. The energy produced by the forcing at the scale k_f^{-1} flows to larger scales, whereas the enstrophy produced by the forcing at the same scale flows to smaller scales [6–8]. Thus two cascades are formed: the energy cascade (inverse cascade) realized at scales larger than the forcing scale k_f^{-1}, and the enstrophy cascade realized at scales smaller than the forcing scale k_f^{-1}. In an unbound system, the energy cascade is terminated by the bottom friction at the scale

$$L_\alpha = \epsilon^{1/2}\alpha^{-3/2}, \tag{2}$$

where a balance between the energy flux ϵ and the bottom friction is achieved. The enstrophy cascade is terminated by viscosity (or hyperviscosity) [3].

If the box size L is larger than L_α, then the above two-cascade picture is realized. We consider the opposite case $L < L_\alpha$. Then the energy, transferred to the box size L by the inverse cascade, is accumulated there, giving rise to a mean (coherent) flow. We consider the statistically stationary case in which the mean flow is already formed. To describe the flow, we use the Reynolds decomposition, that is, the flow velocity is presented as the sum $\boldsymbol{V} + \boldsymbol{v}$, where $\boldsymbol{V}$ is the mean velocity and $\boldsymbol{v}$ represents velocity fluctuations in the background of $\boldsymbol{V}$. Let us stress that $\boldsymbol{V}$ is an average over time, whereas it possesses a complicated spatial structure.

As numerics and experiment show, the coherent flow contains some vortices separated by a hyperbolic flow. The characteristic velocity V of the coherent motion can be estimated as $V \sim \sqrt{\epsilon/\alpha}$. The estimate is a simple consequence of the energy balance: the incoming energy rate ϵ has to be compensated by the bottom friction rate. The characteristic mean vorticity in the hyperbolic region is estimated as $\Omega \sim L^{-1}\sqrt{\epsilon/\alpha}$. However, inside the coherent vortices, the mean vorticity Ω is much higher.

Due to the energy accumulation at the box size, the average flow appears to be much stronger than the flow fluctuations. Therefore, in the main approximation one can neglect fluctuations, dissipation (bottom friction and viscosity), and pumping to proceed to the Euler equation for the mean flow. Since we consider the stationary case, we end up with

$$\boldsymbol{V}\nabla\Omega = 0, \tag{3}$$

where $\Omega = \mathrm{curl}\boldsymbol{V}$. Equation (3) is valid (in the main approximation) both inside the vortices and in the hyperbolic region.

III. COHERENT VORTEX

Here we examine the flow inside the coherent vortex. We attach the origin of our reference system to the vortex center that is determined as the point of maximum vorticity. The definition corresponds to the procedures used in Refs. [13–15] to establish the mean vortex profile. The position of the vortex center fluctuates; for the laboratory experiments, it fluctuates near a fixed position determined by the cell geometry. For the periodic setup (used in the numerics), the vortex center can shift essentially from its initial position, and only the average relative position of the vortices is fixed. The reference system is not inertial, and the velocity of the vortex center is subtracted from the flow velocity in the system. However, the flow vorticity in the reference system coincides with the one in the laboratory reference system.

As was established in Refs. [13–15], in the chosen reference system the mean flow possesses axial symmetry. Such flow can be characterized by the polar velocity that is U-dependent on the distance r from the vortex center. Then the average vorticity is $\Omega = \partial_r U + U/r$. Obviously, such isotropic flow satisfies Eq. (3) since the derivative $\nabla\Omega$ is directed along the radius vector $\boldsymbol{r}$, whereas the velocity $\boldsymbol{V}$ is orthogonal to the radius vector.

Therefore, to obtain an equation for U, one has to use the complete Navier-Stokes equation. Assuming that the average pumping force is zero, one obtains after averaging the Reynolds equation [16]

$$\alpha U = -\left(\partial_r + \frac{2}{r}\right)\langle uv\rangle + \nu\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)U, \tag{4}$$

where v and u are radial and polar components of the velocity fluctuations, and angular brackets denote time-averaging. The average $\langle uv\rangle$ is the Reynolds shear stress. Therefore, to find the r dependence of U, one has to establish the statistics of the flow fluctuations.

To analyze the fluctuations, it is convenient to use the equation for the fluctuating vorticity ϖ,

$$\partial_t\varpi + (U/r)\partial_\varphi\varpi + v\partial_r\Omega + \nabla(\boldsymbol{v}\varpi - \langle\boldsymbol{v}\varpi\rangle) = \phi - \hat{\Gamma}\varpi, \tag{5}$$

which is obtained from the Navier-Stokes equation. Here φ is a polar angle, ϕ is a curl of the pumping force, $\boldsymbol{v}$ is the fluctuating velocity, and the operator $\hat{\Gamma}$ presents dissipation including both the bottom friction and the viscosity term, $\hat{\Gamma} = \alpha - \nu\nabla^2$. For the case of hyperviscosity, the last contribution to $\hat{\Gamma}$ should be modified; it is substituted by $(-1)^{p+1}\nu_p(\nabla^2)^p$. After solving Eq. (5), one can restore the velocity from the relation $\varpi = \partial_r v + v/r - \partial_\varphi u/r$ and the incompressibility condition $\partial_r u + u/r + \partial_\varphi v/r = 0$ with the boundary condition $u,v = 0$ at $r = 0$.

Now we specify the statistical properties of the pumping ϕ. They are assumed to be isotropic and invariant under $\phi \to -\phi$ and time inversion. In other words, odd correlation functions are zero and even correlation functions of ϕ are invariant under $t \to -t$ and space rotations. An example of such pumping is analyzed in the Appendix, where ϕ is assumed to be briefly correlated and possessing Gaussian statistics, which are characterized by the pair correlation function depending on the distance between the points. This case can be analyzed in detail (provided the flow fluctuations are passive); see the Appendix. However, some conclusions can be drawn from symmetry reasoning.

The left-hand side of Eq. (5) changes its sign under the combined transformation

$$t \to -t,\ \varpi \to \varpi,\ \varphi \to -\varphi,\ r \to r,\ v \to -v,\ u \to u. \tag{6}$$

That is why in the case $\hat{\Gamma} = 0$, correlation functions of the velocity fluctuations have to be invariant under the transformation (6). Therefore, the average $\langle uv\rangle$ or the average $\langle u\varpi\rangle$ should be zero, since the averages change sign at the transformation (6). However, in the case $\hat{\Gamma} = 0$ the system is inhomogeneous in time. To ensure the homogeneity, one should take a finite $\hat{\Gamma}$, which makes the averages nonzero. They remain finite in the limit $\hat{\Gamma} \to 0$, which is a manifestation of the dissipation anomaly that is well known in turbulence. One of the manifestations of the anomaly is the so-called d'Alembert paradox.

A. Universal interval

The viscous core is in the center of the coherent vortex. As we are interested in the region outside the core, we neglect the viscous term in Eq. (4), staying with

$$\alpha U = -\left(\partial_r + \frac{2}{r}\right)\langle uv\rangle = -\langle u\varpi\rangle. \tag{7}$$

The last relation in Eq. (7) can be checked using the isotropy (φ independence) of the averages. As follows from Eq. (7), the main goal of our calculations is the average $\langle uv\rangle$ or the average $\langle u\varpi\rangle$.

Furthermore, we consider the region outside the vortex core where the coherent velocity gradient is large enough,

$$U/r \gg \epsilon^{1/3}k_f^{2/3}. \tag{8}$$

In this case, fluctuations in the interval of scales between the pumping scale k_f^{-1} and the radius r are strongly suppressed by the coherent flow. The inequality (8) means that the average velocity gradient U/r is larger than the gradient of the velocity fluctuations in the interval of scales. Therefore, the passive regime is realized there, i.e., the self-interaction of the velocity fluctuations is negligible.

Moreover, the passive regime is then realized for scales smaller than the pumping scale k_f^{-1}. Indeed, in the direct cascade the velocity gradients can be estimated as $\epsilon^{1/3}k_f^{2/3}$, up to logarithmic factors weakly dependent on scale; see [17–19]. Therefore, inequality (8) represents the dominating coherent velocity gradient in the interval of scales where the direct cascade would be realized.

The passive regime can be consistently analyzed. Then one neglects the term in Eq. (5) that is nonlinear in the velocity fluctuations, staying with a linear equation for the vorticity fluctuation ϖ. The equation enables one to express ϖ in terms of the pumping ϕ and then to calculate the correlation functions of ϖ via the correlation functions of ϕ. As we are aiming to calculate the Reynolds shear stress $\langle uv\rangle$, we are interested mainly in the pair correlation function of ϖ.

Furthermore, we focus on the case in which the pumping ϕ is briefly correlated in time and has Gaussian statistics. Direct calculations (see the Appendix) show that in this case,

$$\langle uv\rangle = \epsilon/\Sigma, \tag{9}$$

where Σ is the local shear rate of the coherent flow,

$$\Sigma = r\partial_r(U/r) = \partial_r U - U/r. \tag{10}$$

The expression (9) is derived at the condition $\Sigma \gg \gamma,\alpha$, which is guaranteed by the inequalities (1) and (8). Some additional condition $\gamma \gg \alpha$ is needed to validate expression (9). The inequality $\gamma \gg \alpha$ is assumed to be satisfied in our scheme. (Note that the inequality is satisfied in numerics [15].) The opposite case warrants some additional analysis, but that is beyond the scope of our work.

Substituting expression (9) into Eq. (7), one finds a solution,

$$U = \sqrt{3\epsilon/\alpha}, \qquad \Sigma = -U/r, \tag{11}$$

for the mean profile. Thus we arrive at the flat profile of the polar velocity found in Ref. [15].

Expression (9) is in accordance with our expectation based on symmetry reasoning. Indeed, in the absence of dissipation, the average $\langle uv\rangle$ is zero due to the symmetry of the system under the transformation (6). As we demonstrate in the Appendix, the bottom friction cannot produce nonzero $\langle uv\rangle$. Therefore, its value is related to viscosity (hyperviscosity) and it should be determined by short scales where the viscosity (hyperviscosity) becomes relevant and kills the flow fluctuations. (This is demonstrated explicitly in the Appendix.) That is why Σ is present in the denominator of expression (9) since just Σ determines the dissipation rate at the viscous scale.

The left-hand side of inequality (8) diminishes as r grows. Therefore, it is broken at some $r \sim R_u$. Substituting expression (11) into Eq. (8), one obtains

$$R_u = L_\alpha^{1/3}k_f^{-2/3} = \epsilon^{1/6}\alpha^{-1/2}k_f^{-2/3}. \tag{12}$$

Note that R_u can be larger or smaller than the box size L, depending on the system parameters. The case $R_u > L$ is, probably, characteristic of the numerics [13] and the experiments [14], and the passive regime is then realized everywhere in the box. In contrast, in numerics [15] the universal region is relatively small, $R_u < L$, and is well separated from the outer region, which is not completely passive.

B. Viscous core

The universal profile (11) implies neglecting viscosity in the equation for the average velocity. Since the average velocity (11) is independent of the separation, the mean velocity gradient increases as r diminishes. In that situation, the viscosity is responsible for forming the vortex core. To find the core radius R_c, one can use the equation for the average polar velocity (4). Comparing the left-hand side in Eq. (4) and the viscous term, one finds an estimation for the core radius,

$$R_c \sim (\nu/\alpha)^{1/2}. \tag{13}$$

At $r \ll R_c$, the viscosity dominates and therefore $U \propto r$, which corresponds to a solid rotation.

The estimate (13) can obviously be generalized for the case of hyperviscosity. Taking the dissipation operator in the form $\hat{\Gamma} = (-1)^{p+1}\nu_p(\nabla^2)^p$, one obtains $R_c \sim (\nu_p/\alpha)^{1/(2p)}$ instead of Eq. (13). If $p = 1$, we return to Eq. (13).

The universal profile (11) is realized in the interval of distances $R_c \ll r \ll R_u$. The interval exists if $R_c \ll R_u$. The inequality is equivalent to inequality (1) for $\gamma = \nu k_f^2$, that is, $\epsilon^{1/3}k_f^{2/3} \gg \nu k_f^2$. The inequality means that the characteristic velocity gradient at the pumping scale is much larger than the viscous damping there. In other words, the inequality $R_c \ll r \ll R_u$ is equivalent to one enabling direct cascade in the traditional two-cascade picture. The above arguments are directly generalized for the case of superviscosity.

Note that a derivation of expression (9) implies the inequality $k_f r \gg 1$ justifying the shear approximation for the average velocity (used in the Appendix). Substituting here $r = R_c$, we find $\nu k_f^2 \gg \alpha$. More generally, the inequality $\gamma \gg \alpha$ has to be valid for our analysis of the viscous core. In the opposite case, $\alpha \gg \gamma$, the shear approximation is destroyed at $r \sim k_f^{-1}$, which has to be the lower border of the profile (11).

The structure of the coherent vortices in the case of zero bottom friction, $\alpha = 0$, was established in [20]. In this case, just the viscosity determines the structure.

C. Outer region

Let us consider the region outside the interval with the universal flat profile (11), that is, the case $r > R_u$. We refer to this region as the outer one, and it exists if $R_u < L$. One expects that the average motion remains isotropic there. In the outer region, $\Sigma \ll \epsilon^{1/3} k_f^{2/3}$. Therefore, the passive regime is substituted here by a mixed one. In the interval of scales from k_f^{-1} to $\epsilon^{1/2}/\Sigma^{3/2}$ the traditional inverse cascade is realized, whereas at larger scales the coherent motion modifies the inverse cascade essentially. The direct (enstrophy) cascade is weakly influenced by the coherent flow in the outer region.

The symmetry (6) leads us to the conclusion that the average $\langle uv\rangle$ is formed at the scales where the viscosity (hyperviscosity) comes into play. This property is demonstrated in detail for the passive regime where consistent calculations can be performed (see the Appendix). In the outer region, such consistent calculations cannot be performed. Therefore, conclusions should be based on symmetry reasoning. Thus, we expect that the average $\langle uv\rangle$ can be determined by an expression like (9), where the denominator is merely the characteristic dissipation rate at the viscous (hyperviscous) scale. Based on this, one would expect the expression $\langle uv\rangle \sim \epsilon/(\epsilon^{1/3} k_f^{2/3})$ since $\epsilon^{1/3} k_f^{2/3}$ is just the characteristic dissipation rate in the direct cascade. (Again, to avoid a misunderstanding, note that in the above reasoning we ignored a weak logarithmic dependence of the vorticity correlation functions in the direct cascade; see [17–19].)

However, for the traditional direct cascade, $\langle uv\rangle = 0$ because of the isotropy of the cascade. Therefore, the main contribution to $\langle uv\rangle$, estimated as $\epsilon/(\epsilon^{1/3} k_f^{2/3})$, is absent. The isotropy is weakly broken by the presence of coherent flow, and its influence can be characterized by the dimensionless parameter $\Sigma/(\epsilon^{1/3} k_f^{2/3})$. One expects that the main contribution to the average $\langle uv\rangle$ is linear in Σ. Adding the factor $\Sigma/(\epsilon^{1/3} k_f^{2/3})$ to the above estimate, we find

$$\langle uv\rangle \sim \frac{\epsilon^{1/3}\Sigma}{k_f^{4/3}}. \tag{14}$$

Of course, at $r \sim R_u$, expression (14) becomes expression (9).

Substituting expression (14) into Eq. (7), one obtains

$$\alpha U \sim \frac{\epsilon^{1/3}}{k_f^{4/3}}\left(\partial_r + \frac{2}{r}\right)\left(r\partial_r \frac{U}{r}\right). \tag{15}$$

This expression implies a fast (exponential) decay of U at $r > R_u$ on a length $\sim R_u$. Thus, R_u can be treated as the coherent vortex size, where the mean vorticity is much larger than its typical value $L^{-1}\sqrt{\epsilon/\alpha}$.

One can think about the case $R_u \ll L$. Then, as we demonstrate further, a lot of vortices should appear, separated by the distance $\sim R_u$. Therefore, even in the case $R_u \ll L$, there is no region outside R_u where the mean flow profile is, rigorously, isotropic. Thus the consideration of this subsection is mainly qualitative. Nevertheless, the conclusion about fast attenuation of Ω at $r > R_u$ at the scale $\sim R_u$ remains correct. Note that at $R_u \ll L$, the mean vorticity at the border of the universal region, $\epsilon^{1/2}\alpha^{-1/2}R_u^{-1}$, is still much larger than the typical mean vorticity in the hyperbolic region, $\epsilon^{1/2}\alpha^{-1/2}L^{-1}$. Therefore, a layer of the order R_u, where the vorticity diminishes from $\epsilon^{1/2}\alpha^{-1/2}R_u^{-1}$ to $\epsilon^{1/2}\alpha^{-1/2}L^{-1}$, does exist.

IV. HYPERBOLIC REGION

One can generalize the symmetry reasoning formulated in Sec. III to the case of the hyperbolic region. For that purpose, we introduce the curvilinear reference system related to the Lagrangian trajectories of the mean velocity $\boldsymbol{V}$. In the reference system, the mean velocity has the only component U (along the Lagrangian trajectories). Let us also introduce components of the fluctuating velocity, u and v, longitudinal and transverse to the Lagrangian trajectories. Clearly, inside the coherent vortices the quantities pass to the ones introduced previously.

In terms of the introduced variables, the Euler equation is invariant under the transformation

$$t \to -t,\ \varpi \to \varpi,\ \Omega \to \Omega,\ U,u \to U,u,\ v \to -v. \tag{16}$$

In addition, one should change the sign of the coordinate along the Lagrangian trajectories. Clearly, the transformation (16) is a direct generalization of the transformation (6) (relevant for the interior of the coherent vortices).

An invariance of the Euler equation and of the pumping statistics under the transformation (16) means that the average $\langle uv\rangle$ or $\langle u\varpi\rangle$ is formed at the dissipation scale. Therefore, we can use the same estimate (14) for the average $\langle uv\rangle$ for the hyperbolic region as well. Substituting there $\Sigma \sim L^{-1}\sqrt{\epsilon/\alpha}$, one obtains

$$\langle u\varpi\rangle \sim \sqrt{\epsilon\alpha}\,\frac{R_u^2}{L^2}, \tag{17}$$

where we exploited the fact that the characteristic scale of the coherent flow is L.

Now we examine an equation for the coherent flow that can be obtained by averaging the Navier-Stokes equation. In the principal approximation, the mean velocity $\boldsymbol{V}$ should satisfy Eq. (3). We designate its solution as $\boldsymbol{V}_0$. Due to the dissipation terms and the nonlinear term related to the flow fluctuations, there is a correction $\boldsymbol{V}_1$ to the velocity $\boldsymbol{V}_0$ that satisfies the following equation:

$$\alpha\Omega_0 + (\boldsymbol{V}_0\nabla)\Omega_1 + (\boldsymbol{V}_1\nabla)\Omega_0 + \nabla\langle \boldsymbol{v}\varpi\rangle = 0. \tag{18}$$

Here we omitted the dissipation term (that is irrelevant outside the vortex cores), and we took into account that the average force (exciting the turbulence) is equal to zero.

Multiplying Eq. (18) by Ω_0 and integrating the result over the whole box, one finds

$$\alpha\int dS\,\Omega_0^2 - \int dS\,\nabla\Omega_0\langle \boldsymbol{v}\varpi\rangle = 0. \tag{19}$$

Here we utilized Eq. (3), $\boldsymbol{V}_0\nabla\Omega_0 = 0$, the incompressibility condition, and the boundary conditions. Relation (19) is correct for both the periodic setup and the case of a box with zero velocity at the boundaries. It is a manifestation of the existence of a zero mode of Eq. (3) that can be obtained by multiplying Ω_0 (or $\boldsymbol{V}_0$) by a constant. As it follows from Eq. (3), $\nabla\Omega_0$ is perpendicular to $\boldsymbol{V}_0$. Therefore, one can rewrite Eq. (19) as

$$\alpha \int dS\, \Omega_0^2 = \int dS |\nabla\Omega_0| \langle v\varpi \rangle. \tag{20}$$

It is satisfied in the main approximation in the fluctuation weakness.

Now we can substitute into relation (20) the estimates $\Omega_0 \sim L^{-1}\sqrt{\epsilon/\alpha}$ and $|\nabla\Omega_0| \sim L^{-2}\sqrt{\epsilon/\alpha}$. Then we conclude that the relation cannot be satisfied if $R_u \ll L$. Therefore, we expect that in the case $R_u \ll L$, a number of coherent vortices should appear in the box separated by a distance $\sim R_u$. As far as we can tell, that is the only way to overcome the discrepancy associated with relation (20) and estimate (17).

V. CONCLUSION

We investigated analytically the coherent flow generated by the inverse cascade in a restricted box consisting of a number of vortices and a hyperbolic flow between them. The mean velocity can be estimated as $\sqrt{\epsilon/\alpha}$ (where ϵ is the energy production rate and α is the bottom friction coefficient) everywhere. In addition, the mean vorticity inside the vortices is much larger than in the hyperbolic region, which can be estimated as $\sqrt{\epsilon/\alpha}/L$. The flow inside the vortices is complicated. It can be divided into the following regions: the viscous core, the universal interval, and the outer region. In the universal interval, the velocity fluctuations are passive and the polar mean velocity of the flow is characterized by the flat profile $U = \sqrt{3\epsilon/\alpha}$.

We established the values of the viscous core radius R_c (13) and of the vortex size R_u (12). Subsequent analysis shows that the coherent vortex vorticity diminishes fast in the outer region $r > R_u$; the characteristic length of this decay is $\sim R_u$. Our analysis of the outer vortex region $r > R_u$ is based on symmetry reasoning. This reasoning is confirmed by consistent calculations performed for the passive regime realized at $r < R_u$. A relation between the box size L and the vortex radius R_u can be arbitrary. The case $R_u > L$ is realized in the numerics [13] and experiments [14], and the vortex is not clearly separated from the hyperbolic region. In this case, the passive regime is realized everywhere. In contrast, in numerics [15] the universal region is relatively small, $R_u < L$, and is well separated from the hyperbolic region, where the fluctuations are not completely passive.

In numerics [13,15], dealing with the periodic setup, two coherent vortices were observed, while in experiments [14] a few vortices were observed as well. In this case, the hyperbolic coherent motion is characterized by a scale determined by the box size L. Vortices are then placed in the stagnation points of the hyperbolic flow (up to fluctuations). However, in the limit $R_u \ll L$, we expect appearing a lot of vortices with complicated hyperbolic flow between them; see Sec. IV. These vortices will most likely arrange as a lattice. A natural space structure for the lattice is a chessboard. (A similar lattice was observed in [11].) However, other possibilities (say, the hexagonal or the honeycomb structure) are not excluded.

We formulated conditions under which the coherent vortices with the universal profile (11) appear in 2D turbulence in a finite box. One of the conditions is that the viscous (hyperviscous) dissipation at the pumping scale k_f^{-1} should be much weaker than the characteristic velocity gradient $\epsilon^{1/3}k_f^{2/3}$. This condition was violated in early simulations [10–12] (to extend the interval for the inverse cascade). If $\epsilon^{1/3}k_f^{2/3}$ is of the order of the viscous damping at the pumping scale, then $R_u \sim R_c$ and the universal interval is absent. That is why the universal profile (11) was not observed in the works.

Note also that the inverse energy cascade is observed for surface solenoidal turbulence excited by waves caused by Faraday instability [21,22]. It would be interesting to extend our analysis to this case. This will be a subject of future investigations.

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APPENDIX: PASSIVE REGIME

Here we investigate the passive regime of flow fluctuations in the background of the coherent isotropic vortex characterized by the average polar velocity $U(r)$. In this case, consistent calculations can be done. As we already noted in the main body of the paper, the passiveness implies inequality (8), which guarantees the weakness of the interaction of the flow fluctuations at all scales. Neglecting the nonlinear terms (which are responsible for the interaction) in Eq. (5), one obtains the linear equation for the fluctuating vorticity,

$$\partial_t\varpi + (U/r)\partial_\varphi\varpi + v\partial_r\Omega = \phi - \hat{\Gamma}\varpi. \tag{A1}$$

Equation (A1) describes the dynamics of the flow fluctuations in the background of the average (coherent) flow.

The dissipation in Eq. (A1) (the last term in the equation) is caused by both the bottom friction and the viscosity (or, in numerics, by the hyperviscosity). Therefore, in a Fourier representation the operator $\hat{\Gamma}$ can be written as

$$\Gamma(k) = \alpha + \gamma(k/k_f)^{2p}, \tag{A2}$$

where k is a wave vector and k_f is the characteristic wave vector of the pumping force. For the viscosity $p = 1$, while for the hyperviscosity $p > 1$. Due to the inequalities (1) and (8), the inequality $\Sigma \gg \alpha,\gamma$ is satisfied, where Σ is the local shear rate of the coherent flow defined by Eq. (10).

Below we assume the inequality $k_f r \gg 1$. As we explained in the main body of the paper, in the whole region of the existence of the universal profile (11), $R_c < r < R_u$, the condition $k_f r \gg 1$ is guaranteed by the inequality $\alpha \ll \gamma$. The smallness of the pumping scale k_f^{-1} in comparison with the radius r enables one to proceed to the shear approximation for the vorticity fluctuations, which is correct in leading order in $(k_f r)^{-1}$. The effective shear rate is determined by Eq. (10).

Let us consider the fluctuation dynamics near a circle of radius r_0. Passing to the reference system rotating with angular velocity Ω_0 (corresponding to the radius r_0), one finds from Eq. (A1) in leading order

$$\partial_t \varpi + \Sigma x_1 \frac{\partial \varpi}{\partial x_2} + \hat{\Gamma}\varpi = \phi, \tag{A3}$$

where Σ is taken at $r = r_0$, $x_1 = r - r_0$ is the radial coordinate, and $x_2 = r_0\varphi$ is the angular coordinate. Equation (A3) describes the passive evolution of the flow fluctuations in the shear flow with the shear rate Σ.

Equation (A3) leads to the homogeneous statistical properties of the flow fluctuations in the space (x_1,x_2). Therefore, it is worthwhile to perform a Fourier transform over x_1 and x_2. Rewriting the evolution equation (A3) for the spatial Fourier component of the vorticity $\varpi_{\boldsymbol k}$, one obtains

$$\partial_t \varpi_{\boldsymbol k} - \Sigma k_2 \partial \varpi_{\boldsymbol k}/\partial k_1 + \Gamma(k)\varpi_{\boldsymbol k} = \phi_{\boldsymbol k}. \tag{A4}$$

A formal solution of Eq. (A4) is written as

$$\varpi_{\boldsymbol k}(t) = \int^t d\tau\, \phi[\tau, k_1 + (t-\tau)\Sigma k_2, k_2] \times \exp\left\{-\int_\tau^t d\tau' \Gamma\left(\sqrt{[k_1 + (t-\tau')\Sigma k_2]^2 + k_2^2}\right)\right\}, \tag{A5}$$

where the integral is taken over the time interval where the pumping is switched on.

Furthermore, we assume that the pumping is shortly correlated in time. Therefore, it is characterized by the pair correlation function that can be written as

$$\langle \phi_{\boldsymbol k}(t)\phi_{\boldsymbol q}(t')\rangle = 2(2\pi)^2 \epsilon \delta(\boldsymbol k + \boldsymbol q)\delta(t-t')k^2\chi(\boldsymbol k) \tag{A6}$$

in the Fourier representation. The function $\chi(\boldsymbol k)$ is concentrated in the vicinity of k_f, and it has to be normalized as

$$\int \frac{d^2\boldsymbol k}{(2\pi)^2}\chi(\boldsymbol k) = 1 \tag{A7}$$

to provide the energy pumping rate ϵ. We also assume isotropy of pumping. In other words, the function $\chi(\boldsymbol k)$ depends solely on k, i.e., the absolute value of $\boldsymbol k$.

One finds directly from expressions (A5) and (A6) the simultaneous pair correlation function of the vorticity,

$$\begin{aligned}\langle \varpi_{\boldsymbol k}(t)\varpi_{\boldsymbol q}(t)\rangle &= 2(2\pi)^2\epsilon\delta(\boldsymbol k + \boldsymbol q) \\ &\times \int_0^T d\tau \left[(k_1 + \Sigma\tau k_2)^2 + k_2^2\right]\chi(k_1 + \Sigma\tau k_2, k_2) \\ &\times \exp\left\{-2\int_0^\tau d\tau'\, \Gamma\left(\sqrt{[k_1 + \Sigma\tau' k_2]^2 + k_2^2}\right)\right\},\end{aligned} \tag{A8}$$

where T is the duration of the period before t when the pumping was switched on. In the stationary case, one should take the limit $T \to \infty$. We are interested only in this stationary case.

Using the relation $v_{\alpha\boldsymbol k} = i\epsilon_{\alpha\beta}(k_\beta/k^2)\varpi_{\boldsymbol k}$, we can calculate the correlation functions of the velocity fluctuations in the Fourier representation. Performing the inverse Fourier transform, we find the velocity correlation function. Below, we concentrate on the single-point correlation function $\langle uv\rangle$, which is written as

$$\begin{aligned}\langle uv\rangle = -2\epsilon \int_0^T d\tau \int \frac{d^2k}{(2\pi)^2}\frac{k_1k_2}{\left(k_1^2 + k_2^2\right)^2} &\times \left[(k_1 + \Sigma\tau k_2)^2 + k_2^2\right]\chi(k_1 + \Sigma\tau k_2, k_2) \\ &\times \exp\left\{-2\int_0^\tau d\tau'\, \Gamma\left(\sqrt{[k_1 + \Sigma\tau' k_2]^2 + k_2^2}\right)\right\}.\end{aligned} \tag{A9}$$

Furthermore, we proceed to the variable $\boldsymbol q = (k_1 + \Sigma\tau k_2, k_2)$.

We first consider the case $\Gamma(k) = 0$. Then the passive equation (A1) is invariant under the transformation (6) as well as the complete equation (5). That is why the integral

$$-2\epsilon \int_0^T d\tau \int \frac{d^2q\, q^2\chi(\boldsymbol q)}{(2\pi)^2}\frac{(q_1 - \Sigma\tau q_2)q_2}{\left[(q_1 - \Sigma\tau q_2)^2 + q_2^2\right]^2}, \tag{A10}$$

determining $\langle uv\rangle$ at $\Gamma = 0$, is zero at any finite T. Indeed, the average $\langle uv\rangle$ changes its sign at the transformation (6) and should be equal to zero at $\Gamma = 0$. Introducing a finite Γ breaks the symmetry and makes the average $\langle uv\rangle$ nonzero.

Furthermore, we analyze the stationary case (implying $\Gamma \neq 0$), and we take the limit $T \to \infty$. Passing to the variable $\boldsymbol q = (k_1 + \Sigma\tau k_2, k_2)$ and taking an integral in part, one finds from Eq. (A9)

$$\langle uv\rangle \equiv \langle v_1v_2\rangle = \frac{\epsilon}{\Sigma}(1 - Q), \tag{A11}$$

where Q is defined as

$$\begin{aligned}Q = 2\int_0^\infty d\tau \int \frac{d^2\boldsymbol q\, q^2\chi(\boldsymbol q)}{(2\pi)^2}\frac{\Gamma\left(\sqrt{(q_1 - \Sigma\tau q_2)^2 + q_2^2}\right)}{(q_1 - \Sigma\tau q_2)^2 + q_2^2} \\ \times \exp\left[-2\int_0^\tau d\tau'\, \Gamma\left(\sqrt{(q_1 - \Sigma\tau' q_2)^2 + q_2^2}\right)\right].\end{aligned} \tag{A12}$$

This expression is a starting point of subsequent analysis.

Let us analyze the case in which the viscosity (hyperviscosity) is stronger than the bottom friction at the pumping scale k_f^{-1}, $\gamma \gg \alpha$. Then α can be neglected in the expression (A12) and we stay with $\Gamma(k) = \gamma(k/k_f)^{2p}$. An inspection of expression (A12) shows that for $p > 1/2$, both q_1 and q_2 are of the order of k_f. The estimate implies that $\chi(\boldsymbol q)$ decreases fast enough as q grows. There is a potentially dominant contribution to the value of Q from the region $q_2 \ll k_f$. However, at $p > 1/2$ the contribution gained from small q_2 is smaller than that gained from $q_2 \sim k_f$. If $q_1,q_2 \sim k_f$, then for a characteristic τ the inequality $\Sigma\tau \gg 1$ is satisfied. Then we easily obtain the estimate

$$Q \sim \left(\frac{\gamma}{\Sigma}\right)^{2/(2p+1)} \ll 1. \tag{A13}$$

Thus, we arrive at the conclusion that if viscosity (or hyperviscosity) dominates over α at the pumping scale k_f^{-1}, then $Q \ll 1$ in the passive regime. Therefore, expression (A11) leads to expression (9).

Note that the characteristic time τ_0 in expression (A9) or in expression (A12) is determined by the condition

$\tau_0 \Gamma(k_f \Sigma \tau_0) \sim 1$, that is, $\gamma \tau_0 (\Sigma \tau_0)^{2p} \sim 1$. The relation determines the characteristic time that is needed to enhance (by the influence of the coherent flow) the wave vector $\boldsymbol{q}$ from the initial value $\sim k_f$ to the value $\sim \tau_0 \Sigma k_f$. The enhancement is caused by the deformation of the flow fluctuations (produced by pumping) in the coherent shear flow. At $\tau \sim \tau_0$, dissipation comes into play and stops the flow fluctuations. We conclude that the main contribution to the average $\langle uv \rangle$ is determined by small scales (large wave vectors) where dissipation is relevant. This conclusion is in accordance with our symmetry arguments based on transformation (6).

It is instructive to consider the case $\Gamma = \alpha$, that is, $\gamma = 0$. In this case, $Q = 1$. This can be explicitly obtained from expression (A12) after integrating over the polar angle in q space and using condition (A7). Let us stress that this property exploits the isotropy of the pumping statistics, that is, the assumption that χ is a function of the absolute value of $\boldsymbol{q}$. Thus, in this case $\langle uv \rangle = 0$. Therefore, although the bottom friction breaks the symmetry under the transformation (6), it cannot produce a nonzero value of $\langle uv \rangle = 0$. One expects that at $\alpha \gg \gamma$, the difference $1 - Q$ is small in the parameter γ/α. Therefore, expression (9) is not correct, and some additional analysis is needed that is beyond the scope of our work.

Let us consider in more detail the mean velocity inside the viscous core. We start from Eq. (4), where expression (9) has to be substituted. Furthermore, we assume that R_c is larger than the pumping scale k_f^{-1}, $k_f R_c \gg 1$, and we analyze the region $k_f r \gg 1$, where

$$\langle v\varpi \rangle = \left(\partial_r + \frac{2}{r}\right) \frac{\epsilon}{r \partial_r (U/r)}. \tag{A14}$$

Then the mean profile can be written as

$$U(r) = \sqrt{\frac{\epsilon}{\alpha}} f\left(\frac{r}{R_c}\right). \tag{A15}$$

Equation (4) is then reduced to an equation for the dimensionless function $f(\rho)$:

$$f'' + \frac{1}{\rho} + \left(1 - \frac{1}{\rho^2}\right) + \left(\partial_\rho + \frac{2}{\rho}\right) \frac{1}{f' - f/\rho} = 0. \tag{A16}$$

The boundary conditions for Eq. (A16) are an absence of singularity at $\rho \to 0$ and the asymptotic behavior $f \to \sqrt{3}$ at $\rho \to \infty$.

One can find an expansion of the function $f(\rho)$ at small ρ, that is,

$$f(\rho) = A\rho + B\rho^3 + C\rho^7 + \cdots, \tag{A17}$$

where the fifth-order term is absent. The factor C and coefficients at higher terms are expressed via A and B, say, $C = B^2(2B + A)/6$. To establish the coefficients A and B, one should impose two conditions: an absence of an exponentially growing contribution to f at large ρ, and matching to the region $\rho \lesssim k_f R_c$ where Eq. (A16) has to be modified. This situation is typical of a boundary layer.

[1] A. A. Townsend, *The Structure of Turbulent Shear Flow* (Cambridge University Press, Cambridge, 1976).
[2] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[3] G. Boffetta and R. E. Ecke, Annu. Rev. Fluid Mech. **44**, 427 (2012).
[4] K. Avila *et al.*, Science **333**, 192 (2011).
[5] M. H. Buschmann and V. Gad-El-Hak, AIAA J. **41**, 40 (2003).
[6] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[7] C. E. Leith, Phys. Fluids **11**, 671 (1968).
[8] G. K. Batchelor, Phys. Fluids **12**, 233 (1969).
[9] J. Sommeria, J. Fluid Mech. **170**, 139 (1986).
[10] L. M. Smith and V. Yakhot, Phys. Rev. Lett. **71**, 352 (1993).
[11] L. M. Smith and V. Yakhot, J. Fluid Mech. **274**, 115 (1994).
[12] V. Borue, Phys. Rev. Lett. **72**, 1475 (1994).
[13] M. Chertkov, C. Connaughton, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **99**, 084501 (2007).
[14] H. Xia, M. Shats, and G. Falkovich, Phys. Fluids **21**, 125101 (2009).
[15] J. Laurie, G. Boffetta, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **113**, 254503 (2014).
[16] A. S. Monin and A. M. Yaglom, in *Statistical Fluid Mechanics, Vol. 1: Mechanics of Turbulence*, edited by J. L. Limley (Dover, Mineola, NY, 1971), Chap. 3.
[17] R. H. Kraichnan, J. Fluid Mech. **47**, 525 (1971); **67**, 155 (1975).
[18] G. Falkovich and V. Lebedev, Phys. Rev. E **49**, R1800(R) (1994); **50**, 3883 (1994).
[19] G. Falkovich and V. Lebedev, Phys. Rev. E **83**, 045301(R) (2011).
[20] I. V. Kolokolov and V. V. Lebedev, Pis'ma Zh. Eksp. Teor. Fiz. **101**, 181 (2015) [JETP Lett. **101**, 164 (2015)].
[21] A. von Kameke, F. Huhn, G. Fernández-García, A. P. Muñuzuri, and V. Pérez-Muñuzuri, Phys. Rev. Lett. **107**, 074502 (2011).
[22] N. Francois, H. Xia, H. Punzmann, S. Ramsden, and M. Shats, Phys. Rev. X **4**, 021021 (2014).