IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Kinematic magnetic dynamo in a random flow with strong average shear

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2010 J. Phys. A: Math. Theor. 43 182001

(http://iopscience.iop.org/1751-8121/43/18/182001)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 77.236.39.2
The article was downloaded on 19/04/2010 at 12:43

Please note that terms and conditions apply.

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 182001 (12pp) doi:10.1088/1751-8113/43/18/182001

FAST TRACK COMMUNICATION

# Kinematic magnetic dynamo in a random flow with strong average shear

**V R Kogan[1,2], I V Kolokolov[1,2] and V V Lebedev[1,2]**

[1] Landau Institute for Theoretical Physics RAS, 119334 Kosygina 2, Moscow, Russia
[2] Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Moscow, Russia

E-mail: igor.kolokolov@gmail.com

Received 17 February 2010, in final form 27 March 2010
Published 16 April 2010
Online at stacks.iop.org/JPhysA/43/182001

**Abstract**
We analyze the kinematic dynamo in a conducting fluid where the stationary shear flow is accompanied by relatively weak random velocity fluctuations. The diffusionless and diffusion regimes are described. The growth rates of the magnetic field moments are related to the statistical characteristics of the flow describing the divergence of Lagrangian trajectories. A degree of anisotropy of the magnetic field is estimated. We demonstrate that Zeldovich's 'antidynamo theorem' is wrong.

PACS numbers: 47.35.Tv, 47.65.−d, 94.05.Lk, 95.30.Qd

## 1. Introduction

The dynamo effect is the magnetic field generation by hydrodynamic motions in a conducting medium. We investigate the effect in a conducting fluid (plasma) experiencing a random flow. The principal example of such a flow is the hydrodynamic turbulence [1, 2] responsible for the magnetic field generation in different geophysical and astrophysical phenomena [3–10]. We consider the case where the magnetic field grows from small initial fluctuations and examine the evolution stage when the magnetic field is weak enough, so that one can ignore the feedback from the magnetic field to the flow. The stage where the flow is independent of the magnetic field is called kinematic. The kinematic approximation becomes invalid when the increasing magnetic field starts to affect the fluid motions essentially. In this case the velocity field is strongly influenced by the Lorentz force, so that the induction dynamics is no longer linear in the magnetic field. In most cases that leads to the saturation of the magnetic field and its fluctuations maintained by the hydrodynamic flow. Though the magnetic field cannot be described by a linear equation in the regime, the kinematic stage produces magnetic structures similar to those occurring at the saturation state [11].

We assume that the random flow is statistically homogeneous in space and time. Usually an additional assumption is made that the flow is statistically isotropic. If the velocity field is short-correlated in time, then it is possible to derive closed equations for the magnetic induction

© 2010 IOP Publishing Ltd Printed in the UK & the USA

correlation functions (see, e.g. [12]). The pair correlation function has been analyzed in [13, 14]. The complete statistical description of the magnetic field for a short-correlated smooth statistically isotropic flow has been made in [15], where growth rates and spatial correlation functions of arbitrary order were found. However, in astrophysical applications shear flows are widespread. Such flows are anisotropic and need a special analysis [16]. Here we examine the case where a steady shear flow is complemented by a relatively weak random component. We focus on the analysis of growth rates of moments of the magnetic field (magnetic induction) and on the degree of its anisotropy. We do not specify further the flow statistics but aim to relate the magnetic statistical characteristics to those of the flow, thus revealing the most universal features of the dynamo effect.

An additional motivation for our research comes from dynamics of polymer solutions that in many respects is similar to magneto-hydrodynamics [17, 18]. Particularly, we have in mind the coil-stretch transition [19] (see also [20, 21]) that is an analog of the dynamo effect. A decade ago the so-called elastic turbulence was discovered [22–24] that is a chaotic motion of polymer solutions; the state can be realized even at small Reynolds numbers. The elastic turbulence is a natural frame for applying the dynamo theory to polymer solutions.

The structure of our paper is as follows. In the second section we present basic equations describing the magnetic field in a conducting fluid and formulate a qualitative picture of the kinematic dynamo. In the third section we formulate the Lagrangian formalism enabling us to relate statistical characteristics of the magnetic field to the statistics of the hydrodynamic flow. The fourth section contains an overview of the properties of the evolution matrix and its relation to the divergence of the Lagrangian trajectories. In the fifth section the results concerning the magnetic induction moments and its anisotropy are presented. In the last section we summarize our achievements and discuss perspectives. In the appendix we present analytical results for the short-correlated flow fluctuations.

## 2. Basic relations

We consider magnetic field in a conducting fluid (plasma) where hydrodynamic motions are excited. Then the dynamics of the magnetic field is governed by the following equation (see, e.g., [25]):

$$\partial_t \boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} + \kappa\nabla^2\boldsymbol{B}. \tag{1}$$

Here $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{v}$ is the flow velocity and $\kappa$ is the magneto-diffusion coefficient, inversely proportional to the electrical conductivity of the medium. The flow is assumed to be incompressible, $\nabla\cdot\boldsymbol{v} = 0$. We also assume that the magneto-diffusion term in equation (1) is small in comparison with those related to the flow. We consider the case where the magnetic field is relatively weak and, therefore, its feedback to the flow is negligible. Then equation (1) is a linear equation describing the magnetic field dynamics in a prescribed velocity field. This regime is called kinematic.

The hydrodynamic motion in the fluid is assumed to be random (turbulent) and the velocity statistics is assumed to be homogeneous in space and time. We examine the magnetic field growth from initial fluctuations distributed statistically homogeneously in space at the initial time $t = 0$. The correlation length of the initial fluctuations $l$ is assumed to be smaller than the velocity correlation length $\eta$. If we consider the hydrodynamic turbulence, then the role of the velocity correlation length is played by the Kolmogorov scale. At scales less than $\eta$ the velocity field $\boldsymbol{v}$ can be treated as smooth. The magnetic growth (dynamo) can be characterized by moments of the magnetic induction that exponentially increase over the time $t$:

$$\langle|\boldsymbol{B}(t)|^{2n}\rangle \propto \exp(\gamma_n t); \tag{2}$$

here the angular brackets mean averaging over space. The exponential character of the growth is related to the statistical homogeneity of the flow in space and time and to smoothness of the flow responsible for the growth. The laws (2) are characteristic of the kinematic dynamo.

Our main goal is to express the growth rates $\gamma_n$ in equation (2) via the statistical characteristics of the flow. The natural measure for the growth rates $\gamma_n$ is the so-called Lyapunov exponent of the flow, $\lambda$, that is equal to the average logarithmic divergence rate of close fluid particles. A special question concerns the $n$-dependence of $\gamma_n$. If the magnetic induction statistics is Gaussian, then $\gamma_n \propto n$. Deviations from the linear law signal about an intermittency of the magnetic field. The intermittency implies that the high moments of the magnetic field are determined by rare strong fluctuations.

There are two different regimes of the kinematic magnetic field growth. The first regime is realized if all characteristic scales of the magnetic field are much larger than the magnetic diffusion length $r_d$, $r_d = \sqrt{\kappa/\lambda}$. The assumed smallness of the diffusion coefficient implies the inequality $\eta \gg r_d$. If $l \gg r_d$, then the diffusion term in equation (1) is negligible at the first stage that we call diffusionless. The magnetic force lines are deformed by the flow without reconnections in this regime. However, distortions of the magnetic field by the flow inevitably lead to appearing scales of order $r_d$. After that the magnetic diffusion is switched on that admits reconnections. This second (diffusion) stage is characterized by the growth rates different from the ones describing the diffusionless regime.

Let us present a qualitative picture explaining the magnetic field evolution at the kinematic stage. The initial magnetic field distribution in space can be thought as an ensemble of blobs of sizes $\sim l$. Then the blobs are distorted being stretched in one direction and compressed in another direction. In the isotropic case the directions of stretching and compressing are chaotically varying in space, whereas in our case they are attached to the shear flow: stretching occurs mainly along the shear velocity. At the first (diffusionless) stage blobs are deformed without intersections, and the magnetic field induction grows like a separation between close fluid particles at the stage since equation (1) at $\kappa = 0$ coincides with the equation for a separation of two close fluid particles.

The diffusionless stage finishes when the characteristic blob width diminishes down to the diffusion length $r_d$. Then the diffusion is switched on that leads to two effects. First, the diffusion prevents further shrinking of the blob widths, so that they remain of the order of $r_d$, whereas the blobs continue to be stretched in the direction of the shear velocity. Second, due to reconnections of the magnetic force lines admitted by diffusion the blobs start to overlap. As a result, new blobs of a characteristic longitudinal size $\eta$ are formed, see figure 1. The magnetic induction in such new blobs can be found by averaging the induction of a large number $N$ of initial blobs, the number $N$ grows exponentially over time. Averaging over the large number of random variables leads to the appearance of an exponentially small factor $\sim 1/\sqrt{N}$ in the amplitude of the magnetic induction. Besides, the amplitudes of the initial blobs remain to increase over time as a separation between fluid particles. We conclude that at this second (diffusive) stage the magnetic field is still growing exponentially over time but slower than at the first stage.

We consider the case where the steady shear constituent of the flow is stronger than the random one. Quantitatively, the condition is written as $s \gg \lambda$ where $s$ is the shear rate. Indeed, the Lyapunov exponent in a pure shear flow is zero, and its non-zero value is associated with the presence of the relatively weak random constituent of the flow. The distorted magnetic blobs are elongated mainly along the shear velocity. However, they are tilted with respect to the velocity direction due to the presence of the random velocity component, see figure 1. The tilt possesses the same dynamics as the direction of the polymer stretching in the same flow, see [21]. Therefore, the tilt angle $\theta$, see figure 1, can be estimated as $\theta \sim \lambda/s$. The tilt angle

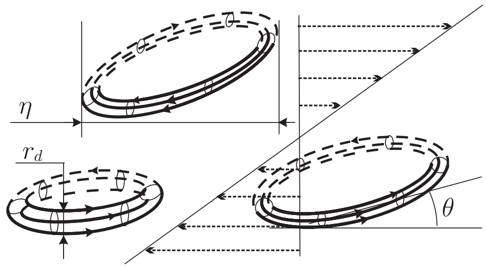


**Figure 1.** Sketch of typical magnetic blobs during the diffusive kinematic stage.

determines the typical ratio of the magnetic field components $B_2/B_1 \sim \lambda/s \ll 1$, where the first axis is directed along the shear velocity that varies along the second axis. Thus, the ratio $s/\lambda$ characterizes an anisotropy degree of the magnetic field.

## 3. Lagrangian dynamics

The magneto-dynamic equation (1) can formally be solved in the framework of the Lagrangian approach to the fluid motion. Passing to the Lagrangian frame, one then finds a formal solution for the induction field $\boldsymbol{B}$ in terms of its initial value $\mathcal{B}$, $\mathcal{B}(\boldsymbol{r}) = \boldsymbol{B}(0, \boldsymbol{r})$. Exploiting a generalization of the scheme proposed in [26, 27] we write the solution as

$$\boldsymbol{B}(\tau, \boldsymbol{r}) = \lfloor \hat{W}(\tau)\mathcal{B}[\boldsymbol{y}(0)]\rfloor, \tag{3}$$

where $\boldsymbol{y}(t)$ is a function of the time $t$ defined over the time interval $0 < t < \tau$ with the boundary condition posed at the final time, $\boldsymbol{y}(\tau) = \boldsymbol{r}$. The function satisfies the following stochastic equation:

$$\partial_t \boldsymbol{y} = \boldsymbol{v}(t, \boldsymbol{y}) + \boldsymbol{\xi}, \tag{4}$$

where $\boldsymbol{\xi}$ is the white noise (Langevin force) with the pair correlation function

$$\langle \xi_i(t_1)\xi_j(t_2)\rangle = 2\kappa\delta_{ij}\delta(t_1 - t_2). \tag{5}$$

The floors in equation (3) mean averaging over the $\xi$ statistics. Note that at $\boldsymbol{\xi} = 0$ equation (4) describes Lagrangian trajectories (trajectories of fluid particles). The noises $\boldsymbol{\xi}$ disturb the trajectories simulating diffusion of the magnetic field. The matrix $\hat{W}(t)$ in equation (3) is determined by the following equation:

$$\partial_t \hat{W} = \hat{\Sigma}\hat{W}, \qquad \hat{W}(0) = 1, \tag{6}$$

where the last term represents the boundary condition. The matrix $\hat{\Sigma}(t)$ is the velocity gradients matrix, $\Sigma_{ji} = \partial_i v_j(t)$, taken at the spacial point $\boldsymbol{y}(t)$. We will call $\hat{W}$ an evolution matrix. One can say that expression (3) describes back in time the evolution of the magnetic field.

In the framework of the formalism, correlation functions of the magnetic field have to be obtained by averaging the product of factors (3) taken at the respective points over the statistics of the noises $\boldsymbol{\xi}$, besides averaging over space. Thus, the moments $\langle|\boldsymbol{B}|^{2n}\rangle$ should be calculated in two steps. First, one should average the product $|\hat{W}\mathcal{B}|^{2n}$ over the $\xi$-statistics given by equation (5), the averaging catches the diffusion effects. (Let us underline that the fields $\boldsymbol{\xi}$ have to be treated as independent for all $2n$ factors in the product.) Second, one should average the result over space that is equivalent to averaging over the velocity statistics. This logic was realized for the isotropic random flow in [15].

The evolution matrix $\hat{W}$ has some universal statistical properties that can be obtained by averaging over Lagrangian trajectories. Our flow is incompressible, that leads to the conclusion that averaging over initial positions of fluid particles is equivalent to averaging over their final (or intermediate) positions. Therefore, averaging over Lagrangian trajectories is equivalent to space averaging. Since the magnetic diffusion is weak one can neglect its influence to the statistical properties of the matrix $\hat{W}$. Postponing a discussion of the statistics (see below) we formulate here only some general assertions concerning eigenvalues of the matrix $\hat{W}(t)$. Its determinant is equal to unity, since the velocity gradient matrix $\hat{\Sigma}$ is traceless, $\mathrm{tr}\,\hat{\Sigma} = 0$, that, in turn, is a consequence of the incompressibility condition $\nabla \cdot \boldsymbol{v} = 0$. Next, the eigenvalues of $\hat{W}$ can be written as $\exp(\rho_1)$, $\exp(\rho_2)$ and $\exp(\rho_3)$ with $\rho_1 + \rho_2 + \rho_3 = 0$. We order the eigenvalues as $\rho_1 > \rho_2 > \rho_3$; then $\rho_1$ is positive whereas $\rho_3$ is negative. One can say that $\rho_1$ is responsible for a forward in time Lagrangian evolution whereas $\rho_3$ is responsible for a back in time Lagrangian evolution. One can estimate $\rho_i$ as $\lambda t$.

In the diffusionless regime, realized at $t \ll \lambda^{-1}\ln(l/r_d)$, one can neglect diffusion effects. Then while calculating the moment $\langle |\boldsymbol{B}|^{2n}\rangle$ one can take a product of the identical factors (3) where $\boldsymbol{y}(t)$ is simply a Lagrangian trajectory terminated at $\boldsymbol{r}$. Then $|\boldsymbol{B}(\boldsymbol{r})|^{2n} \approx \exp(2n\rho_1)|\mathcal{B}|^{2n}$ where $\mathcal{B}$ should be taken at the origin of the Lagrangian trajectory. Here just the factor $\exp(2n\rho_1)$ is responsible for the exponential growth of the moments, and therefore, we can restrict ourselves to the estimation $|\boldsymbol{B}(\boldsymbol{r})|^{2n} \sim \exp(2n\rho_1)\mathcal{B}_0^{2n}$ where $\mathcal{B}_0$ is the characteristic value of the initial magnetic field fluctuations. In the diffusion regime, realized at $t \gg \lambda^{-1}\ln(l/r_d)$, the situation is a bit more complicated.

Let us consider the second moment. Then we should deal with two trajectories, $\boldsymbol{y}$ and $\boldsymbol{y}'$, terminating at the same point $\boldsymbol{r}$ at $t = \tau$, but characterized by independent noises $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$. The second moment is a spacial average of $\boldsymbol{B}^2 = \lfloor\{\hat{W}\mathcal{B}[\boldsymbol{y}(0)]\}\{\hat{W}'\mathcal{B}[\boldsymbol{y}'(0)]\}\rfloor$ (recall that floors mean averaging over the $\xi$-statistics). An appreciable contribution to the second moment is related to the trajectories with $|\boldsymbol{y}(0) - \boldsymbol{y}'(0)| \lesssim l$. Since $|\boldsymbol{y}(0) - \boldsymbol{y}'(0)| \ll \eta$ and $|\boldsymbol{y}(\tau) - \boldsymbol{y}'(\tau)| = 0$, the difference $\Delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}'$ stays to be much less than $\eta$ at any time from the interval $0 < t < \tau$ for such event. Then we obtain from equation (4) expanding the velocity up to linear in $\Delta\boldsymbol{y}$ terms:

$$\partial_t \Delta\boldsymbol{y} = \hat{\Sigma}\Delta\boldsymbol{y} + \boldsymbol{\xi} - \boldsymbol{\xi}', \tag{7}$$

where $\hat{\Sigma}$ is taken at the point $\boldsymbol{y}$. A solution of equation (7), equal to zero at $t = \tau$, is written as

$$\Delta\boldsymbol{y}(t) = -\hat{W}(t)\int_t^\tau \mathrm{d}t_1\,\hat{W}^{-1}(t_1)[\boldsymbol{\xi}(t_1) - \boldsymbol{\xi}'(t_1)]. \tag{8}$$

Since the separation $\Delta\boldsymbol{y}$ is a linear combination of $\boldsymbol{\xi}, \boldsymbol{\xi}'$ it possesses the Gaussian statistics characterized by variances of its components.

We are interested in the $\Delta\boldsymbol{y}$ statistics at $t = 0$. Then the integral in equation (8) is gained at $t_1 \sim \lambda^{-1}$. The different components of $\Delta\boldsymbol{y}$ have different variances; they are estimated as $Y_1 \sim r_d\exp[-\rho_1(\tau)]$, $Y_2 \sim r_d\exp[-\rho_2(\tau)]$ and $Y_3 \sim r_d\exp[-\rho_3(\tau)]$ for the components of $\Delta\boldsymbol{y}$ along eigenvectors of the matrix $\hat{W}^{-1}$. The first variance $Y_1$ is always less than $l$, whereas the third one $Y_3$ is larger than $l$ in the diffusive regime. As to the second variance, $Y_2$, it could be larger or smaller than $l$ depending on the sign of $\rho_2$. Therefore, a probability (in the diffusion regime) for the trajectories $\boldsymbol{y}$ and $\boldsymbol{y}'$ to be closer than $l$ at $t = 0$ can be estimated as $l/Y_3 \sim (l/r_d)\exp[\rho_3(\tau)]$ if $\rho_2 > 0$. Next, we should include the factor $\exp(2\rho_1)$ associated with the matrix $\hat{W}(\tau)$ in equation (3). Finally, for $\rho_2 > 0$ we arrive at

$$\lfloor|\boldsymbol{B}|^2\rfloor \sim \mathcal{B}_0^2(l/r_d)\exp(\rho_1 - \rho_2), \tag{9}$$

where we used the condition $\rho_1 + \rho_2 + \rho_3 = 0$ and $\mathcal{B}_0$ is the characteristic value of the initial magnetic field fluctuations.

The situation with $\rho_2 < 0$ is slightly different. In this case calculation of $\lfloor |\boldsymbol{B}|^2 \rfloor$ is reduced to the integration of $e^{2\rho_1} B_1[\boldsymbol{y}(0)] B_1[\boldsymbol{y}'(0)]$ over $\Delta y_2$ and $\Delta y_3$ with the statistical weight $(Y_2 Y_3)^{-1} \exp\left[-(\Delta y_2)^2 / \left(2Y_2^2\right) - (\Delta y_3)^2 / \left(2Y_3^2\right)\right]$. Typical values of $\Delta y_2$ and $\Delta y_3$ in the integral are estimated as $l$, whereas $Y_2, Y_3 \gg l$. Therefore, in the main approximation the exponent in the integral can be substituted by unity. However, that leads to a zero value of the integral due to the solenoidal nature of the magnetic field $\boldsymbol{B}$. Therefore, one should expand the exponent in $(\Delta y_2)^2 / \left(2Y_2^2\right)$; the first term of the expansion gives $\lfloor |\boldsymbol{B}|^2 \rfloor \sim l^4 Y_2^{-3} Y_3^{-1} e^{2\rho_1} \mathcal{B}_0^2$, that is for $\rho_2 < 0$

$$\lfloor |\boldsymbol{B}|^2 \rfloor \sim \mathcal{B}_0^2 (l/r_d)^4 \exp(\rho_1 + 2\rho_2), \tag{10}$$

where, again, we have used the condition $\rho_1 + \rho_2 + \rho_3 = 0$.

Note that expressions (9) and (10) are equivalent to the ones obtained in the Fourier representation for the isotropic case in [15]. However, expressions (9) and (10), written for real space, are correct for the anisotropic problem as well, and are in fact more convenient for the problem.

Let us turn to the high moments. One can check that a principal contribution to the average $\lfloor |\boldsymbol{B}|^{2n} \rfloor$ is produced by configurations where the $2n$ points $\boldsymbol{y}_\alpha(0)$ are divided into $n$ pairs with separations $\lesssim l$ in each pair. Because of the independence of the white noises $\boldsymbol{\xi}_\alpha$, the probability of such an event can be estimated as a product of probabilities for the second moment, that is

$$\lfloor |\boldsymbol{B}|^{2n} \rfloor \sim \lfloor |\boldsymbol{B}|^2 \rfloor^n, \tag{11}$$

where the second moment is given by equation (9) or equation (10). We have ignored a combinatoric factor in equation (11) being interested in the time dependence of the moments.

At the next step we should average expression (11) over the velocity statistics. Before doing so, we formulate the basic properties of the evolution matrix $\hat{W}$.

## 4. Properties of the evolution matrix

Here we overview the basic properties of the evolution matrix $\hat{W}$ defined by equation (6). Its formal solution is

$$\hat{W}(t) = T \exp\left[\int_0^t \mathrm{d}t' \, \hat{\Sigma}(t')\right], \tag{12}$$

where $T \exp$ means a chronologically ordered exponent. Let us recall that the determinant of the matrix $\hat{W}$ is equal to unity since the matrix $\hat{\Sigma}$ is traceless (due to the incompressibility condition). We are interested in the features of the evolution matrix $\hat{W}$ on times larger than the inverse Lyapunov exponent, $t \gg \lambda^{-1}$, where $\hat{W}$ possesses some universal statistical properties, following from the fact that under the condition the evolution matrix (12) can be treated as a product of a large number of random matrices, see [28–30].

It is convenient for us to use the Gaussian decomposition of the evolution matrix $\hat{W} = \hat{T}_L \hat{\Delta} \hat{T}_R$, where $\hat{T}_L$ and $\hat{T}_R$ are triangle matrices,

$$\hat{T}_L = \begin{pmatrix} 1 & \chi & \chi_1 \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{T}_R = \begin{pmatrix} 1 & 0 & 0 \\ \zeta_1 & 1 & 0 \\ \zeta_2 & \zeta_3 & 1 \end{pmatrix}, \tag{13}$$

$\hat{\Delta}$ is a diagonal matrix and $\hat{\Delta} = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3)$. Since both triangle matrices, $\hat{T}_L$ and $\hat{T}_R$, have unit determinants, the determinant of $\hat{\Delta}$ is equal to unity as well.

Substituting the decomposition $\hat{W} = \hat{T}_L \hat{\Delta} \hat{T}_R$ into the evolution equation (6), one finds

$$\hat{T}_L^{-1} \hat{\Sigma} \hat{T}_L = \hat{T}_L^{-1} \partial_t \hat{T}_L + \partial_t \hat{\Delta} \hat{\Delta}^{-1} + \hat{\Delta} \partial_t \hat{T}_R \hat{T}_R^{-1} \hat{\Delta}^{-1}. \tag{14}$$

The terms in the right-hand side of equation (14) are the left-off-diagonal matrix, the diagonal matrix and the right-off-diagonal matrix, accordingly. Therefore, one obtains a closed nonlinear equation for the matrix $\hat{T}_L$ that leads to a homogeneous in time statistics of the matrix. Next, we obtain for components of the diagonal matrix $\partial_t\hat{\Delta}\hat{\Delta}^{-1}$ expressions that are random variables with the statistics homogeneous in time. Therefore, $\ln\Delta_1$, $\ln\Delta_2$ and $\ln\Delta_3$ are subjects of the central limit theorem. Typically, the variables behave linear in the time $t$ with coefficients of the order of $\lambda$. The situation with the matrix $\hat{T}_R$ is slightly more complicated since there are the exponential factors $\Delta_1, \Delta_2, \Delta_3$ in the last term of equation (14). Therefore, some components of the matrix $\hat{T}_R$ behave exponentially in time like the factors.

For our flow, which is composed of the steady shear flow and the random component, the matrix of the velocity gradients $\Sigma_{ji} = \partial_i v_j$ is a sum of the shear term and the random component:

$$\Sigma_{ji}(t) = s\delta_{j1}\delta_{i2} + \sigma_{ji}(t). \tag{15}$$

Here the first axis of the reference system is directed along the shear velocity that varies along the second axis, and $s$ is the shear rate. The random matrix $\sigma_{ji}$ is zero in average and should be characterized in terms of its correlation functions. Note that the trace of the matrix is zero, $\mathrm{tr}\,\hat{\sigma} = 0$.

Based on the leading role of the shear term in expression (15), one obtains from equation (14) for $\hat{T}_L$ a hierarchy $\chi \gg \chi_1 \gg \chi_2$. Therefore, in the main approximation in $\lambda/s$ the only component, $\sigma_{21}$, is relevant and the equation for $\hat{T}_L$ is reduced to

$$\partial_t\chi = s - \chi^2\sigma, \tag{16}$$

where $\sigma \equiv \sigma_{21}$. We conclude that the variable $\chi$ possesses a homogeneous in time statistics, in accordance with our general expectations. Note that a similar equation was obtained for the tilt angle in [21]. However, our approximation implies $\chi > 0$ whereas the tilt angle can be either positive or negative. Note that $\chi \sim s/\gamma \gg 1$ as follows from equation (16). Keeping the main in $\chi$ contributions to the diagonal terms in equation (14) one obtains $\mathrm{diag}(\partial_t\hat{\Delta}\hat{\Delta}^{-1}) = (-\chi\sigma, \chi\sigma, 0)$. Therefore, in this approximation

$$\mathrm{diag}\,\Delta = (\mathrm{e}^{-\rho}, \mathrm{e}^{\rho}, 1), \tag{17}$$

$$\partial_t\rho = \chi\sigma. \tag{18}$$

If $t \gg \lambda^{-1}$, then typically $\rho \sim \lambda t \gg 1$.

One concludes from the equations for $\zeta_1, \zeta_2, \zeta_3$ following from equation (14) that the variables $\zeta_1$ and $\zeta_2$ are 'frozen' at $t \gg \lambda^{-1}$ at values much less than unity, whereas $\zeta_3 \sim e^{\rho}$. Based on the results obtained for the matrices $\hat{T}_L$, $\hat{\Delta}$ and $\hat{T}_R$ one finds the eigenvalues of the matrix $\hat{W}$. In the main approximation in $\lambda/s$ we obtain $\rho_1 = \rho$, $\rho_2 = 0$, $\rho_3 = -\rho$. Note that the expressions together with equation (18) lead to the relation $\lambda = \langle\chi\sigma\rangle$.

Since the variable $\rho$ is an integral over time of a random quantity with homogeneous in time statistics, see equation (18), the statistics of $\rho$ possesses some universal features at $\lambda t \gg 1$. Namely, under the condition the probability distribution function (PDF) of $\rho$ can be written in a self-similar form [31]

$$P(\rho) \propto \exp[-tS(\rho/t)], \tag{19}$$

where $S$ is the so-called Kramer function (or entropy function). Expression (19) is a manifestation of PDFs for the so-called intensive variables (see, e.g., [32]).

Let us consider moments of the divergence of close Lagrangian trajectories in our random flow. The equation governing a separation between the trajectories $\boldsymbol{R}$ is $\partial_t R_j = \Sigma_{ji} R_i$, that has a solution $\boldsymbol{R}(t) = \hat{W}\boldsymbol{R}(0)$. Therefore, at $t \gg \lambda^{-1}$ we arrive at the estimation $R \sim R_0 e^{\rho}$. Then the moments of $R$ can be calculated in the saddle-point approximation (justified by the inequality $\lambda t \gg 1$):

$$\langle R^n \rangle = \int \mathrm{d}\rho\, P(\rho) R^n \propto \exp(\lambda_n t), \tag{20}$$

$$\lambda_n = -S(x_n) + n x_n, \qquad \text{where} \quad S'(x_n) = n. \tag{21}$$

Thus, the exponents $\lambda_n$ are determined by statistical properties of the flow. Note that the Lyapunov exponent $\lambda$ is expressed via $\lambda_n$ as $\lambda = \mathrm{d}\lambda_n/\mathrm{d}n|_{n=0}$. Therefore, $\lambda_n \sim \lambda$, a separate question concerns the $n$-dependence of $\lambda_n$.

The statistical properties of the stochastic variables $\chi$ and $\rho$ can be analytically established for the case where flow fluctuations are short correlated in time, see the appendix. The solution illustrates a possible dependence of $\lambda_n$ on the system parameters and on $n$.

## 5. Moments of the magnetic field

In our approximation, $\rho_2 = 0$ and, therefore, expression (11) is correct where $\rho_1$ can be substituted for $\rho$, see above. To find a time dependence of the magnetic field moments one has to perform an additional average of expression (11) over space that is equivalent to averaging over the $\rho$ statistics. Thus, the $2n$th moment of the magnetic field induction is written as

$$\langle \boldsymbol{B}^{2n}(t) \rangle = \int \mathrm{d}\rho\; P(\rho) \lfloor \boldsymbol{B}^{2n}(t) \rfloor. \tag{22}$$

One should substitute here $B^{2n} \sim \exp(2n\rho)\mathcal{B}_0^{2n}$ for the diffusionless regime or expression (11) for the diffusion regime. Since in our case $\rho_1 = \rho$ and $\rho_2 = 0$, both, equations (9) and (10), give the same law

$$\lfloor B^{2n} \rfloor \propto \exp(n\rho). \tag{23}$$

Integral (22) can be calculated in the saddle-point approximation, like for moments (20). Substituting into equation (22) expression (23) and function (19) and then performing the calculation one finds $\gamma_n = \lambda_{2n}$ for the diffusionless regime and $\gamma_n = \lambda_n$ for the diffusion regime. Thus, we related the dynamo exponents introduced by equation (2) to the statistical properties of the flow.

The main contribution to the moments $\langle \boldsymbol{B}^{2n}(t) \rangle$ is associated with the component $B_1$ of the magnetic induction directed along the velocity of the shear flow. Let us now analyze moments of the component $B_2$ directed along the gradient of the shear flow, $\langle B_2^{2n} \rangle$. The moments are much smaller than the moments $\langle \boldsymbol{B}^{2n}(t) \rangle$; therefore, their ratio is a measure of the magnetic field anisotropy caused by the strong shear flow. One finds from equations (13) and (17) that $\lfloor B_1^2(t) \rfloor = \chi^2 \lfloor B_2^2(t) \rfloor$. Thus, $\chi$ is a measure of the magnetic field anisotropy and $\chi^{-1}$ is the tilt angle $\theta$ of the magnetic blobs to the shear velocity, see figure 1. Since the variable $\chi$ possesses a homogeneous in time statistics, the factor $\chi^{-2}$ does not produce a difference in growth rates, that is both moments, $\langle B_1^{2n} \rangle$ and $\langle B_2^{2n} \rangle$, are proportional to the same exponent $\exp(\gamma_n t)$. However, the prefactors at the exponents are different.

To find the difference in the prefactors is not enough to know the statistical properties of the variable $\rho$ that determine the exponent. Generally, one should know a mutual probability distribution of the fields $\sigma(t)$ and $\chi(t)$ that is quite complicated object that depends on the details of the flow dynamics. However, the situation is simplified at large $n$; then the moments $\langle B_{1,2}^2(t) \rangle$ are determined by saddle-point solutions in the functional space (instantons), see [33].

In our case, the instantons are quite simple: due to the homogeneity in time, they correspond to the variables $\sigma$ and $\chi$, independent of time. Then we find $s = \chi^2\sigma$ and $\rho = \chi\sigma t$, as follows from equation (18). Comparing the expressions to $\rho = xt$ one finds $\chi = s/x$. Therefore, one obtains for the diffusion case

$$\langle B_1^{2n}\rangle \approx \left(\frac{s}{x_n}\right)^{2n} \langle B_2^{2n}\rangle, \tag{24}$$

where the quantity $x_n$ is determined by equation (21). In the diffusionless case $x_n$ should be substituted by $x_{2n}$ in relation (24). Let us note that due to the assumed inequality $s \gg \lambda$ the moments of $B_1$ are much larger than ones of $B_2$, indeed. Relation (24) is correct for $n \gg 1$. However, it estimates correctly the ratio of the moments even for $n \sim 1$, then $x_n \sim \lambda$.

There is a question concerning moments of the third component of the magnetic induction, $\langle B_3^{2n}\rangle$. To analyze their behavior one should take into account the components of the matrix $\hat{T}_R$, that we ignored at investigating $B_1$ and $B_2$. Then we conclude that the time dependence of $\langle B_3^{2n}\rangle$ is characterized by the same exponents $\exp(\gamma_n t)$ at the diffusion stage. As to prefactors, they depend on the details of the flow statistics and are not universal even at large $n$.

## 6. Discussion

We have analyzed the kinematic dynamo stage when the small-scale fluctuations of the magnetic field grow in a shear flow complemented by a relatively weak random flow. The weakness is characterized by the inequality $s \gg \lambda$ where $s$ is the shear rate and $\lambda$ is the Lyapunov exponent of the flow. The shear makes the flow strongly anisotropic, which, paradoxically, simplifies the analysis of the dynamo phenomenon since a single component of the random velocity gradient appears to be relevant. Probably, the assumption on a small correlation length of the magnetic fluctuations is not crucial for our scheme, since the small scales in the magnetic field are inevitably produced by the hydrodynamic motion. Let us explain the assertion in more detail. We demonstrated that the principal contribution to the magnetic field moments is related to the prehistory when the diffusion does not strongly disperse the trajectories $\boldsymbol{y}_i$ back in time. If the separation is smaller than both the magnetic correlations length and the velocity correlation length, then the exponential dynamo effect is guaranteed. If the initial magnetic correlation length is larger than the velocity correlation length, then just the latter will determine the boundary between the diffusionless stage and the diffusion one.

Let us underline that in the main approximation our problem is reduced to a purely two-dimensional velocity field (with components along the shear velocity and along its gradient). We have proved an existence of the dynamo in this case (that is the exponential grows of the magnetic field moments). The result obviously contradicts the statement of [34–36] that there cannot be the magnetic dynamo in two-dimensional flows. We assert that this statement is wrong and the error of [34–36] is in ignoring the third component $B_3$ of the magnetic induction (perpendicular to the velocity plane). Let us explain the error in more detail for an unrestricted $2d$ linear velocity profile. Then the third component satisfies the passive scalar equation and, consequently, decays exponentially. However, $B_3$ cannot be ignored under the divergentless condition $\nabla\boldsymbol{B} = 0$ since the characteristic scale of the magnetic field along direction of its growth increases faster than the magnetic field itself. One can check that all the terms in $\nabla\boldsymbol{B} = 0$ decay with the same exponent, and therefore the condition $\nabla\boldsymbol{B} = 0$ leads to an effectively divergent in-plane magnetic field. The dynamo effect is not forbidden for such field. A detailed analysis of the discrepancy will be published elsewhere. An existence of the dynamo effect for purely two-dimensional flows is a subject of numerical verification.

We do not specify the statistical properties of the random flow exploring only its smoothness at scales less than the velocity correlation length $\eta$. Then it is possible to relate the exponents $\gamma_n$ characterizing the kinematic dynamo, see equation (2), to intrinsic characteristics of the flow characterizing divergence of close Lagrangian trajectories, see equation (20). It appears that $\gamma_n = \lambda_{2n}$ in the diffusionless regime and $\gamma_n = \lambda_n$ in the diffusion regime. We also related the anisotropy degree of the magnetic field to the same intrinsic characteristics of the flow, see equation (24). Thus, the main features of the magnetic field statistics (including its intermittency) are dictated by the flow statistics. Note that our general scheme does work without essential modifications for the statistically isotropic flows or for the random flows with other types of anisotropy too.

We have formulated the qualitative picture describing the structure of the magnetic field at the kinematic dynamo stage. The picture can be elaborated to establish quantitative properties of the magnetic induction correlation functions; it is the next subject of our investigation. We hope that the structure of the functions will survive in its principal features even at the saturation state. We believe that the strongly anisotropic case characterized by a strong shear is more perspective from this point of view than the isotropic one. The ideology and the analytical approach developed in our work can be expanded to the dynamics of polymer solutions possessing elasticity that is described similarly to the magnetic field. In this way we hope to clarify some aspects of the so-called elastic turbulence [22–24] that are still not explained.

## Acknowledgments

We thank M Chertkov, G Falkovich, and SS Vergeles for helpful discussions. The work was partially supported by RFBR under grant no 09-02-01346-a and by MES of RF under FTP 'Kadry'.

## Appendix. Short correlated random flow

Here we consider a strong steady shear flow complemented by a shortly correlated in time random component. As we have already noted in the main body of the paper that, under the condition $\lambda \ll s$ (that is a manifestation of the weakness of the fluctuations), the only relevant component of the random velocity gradients matrix is $\sigma \equiv \sigma_{21}$. In the short correlated case it can be treated as white noise, that is

$$\langle \sigma(t_1)\sigma(t_2)\rangle = 4D\delta(t_1 - t_2), \tag{A.1}$$

where the factor $D$ characterizes the strength of the noise. The value of the Lyapunov exponent for the short-correlated case is

$$\lambda = \frac{\sqrt{\pi}\, 3^{1/3}}{\Gamma(1/6)} D^{1/3} s^{2/3}, \tag{A.2}$$

which is obtained in [37]. Thus, the condition $\lambda \ll s$ is equivalent to the inequality $D \ll s$.

The next object of our investigation is the quantity $\chi$ statistical properties of which are determined by equation (16). Note that one should consider large positive $\chi$, otherwise equation (16) would be incorrect. Introducing the variable $\zeta = \chi^{-1}$, one derives from equations (16) and (A.1) the following Fokker–Plank equation for the probability distribution function $P(\zeta)$:

$$\partial_t P = \partial_\zeta (s\zeta^2 P) + 2D\partial_\zeta^2 P. \tag{A.3}$$

A stationary solution of the equation is

$$P \propto \exp\left(-\frac{s\zeta^3}{6D}\right). \tag{A.4}$$

We conclude that the characteristic value of $\chi$ is estimated as $(s/D)^{1/3}$ that is much larger than unity at our assumptions.

Let us now find values of the exponents $\lambda_n$. It is hard to do analytically even for the simplest model formulated above. However, it can be easily done for a large $n$ exploiting the saddle-point (instantonic) approximation in the functional space [33]. The approximation leads to the time-independent saddle-point values of $\chi$ and $\sigma$. Then we find from equations (16) and (18) that $\chi = \sqrt{s/\sigma}$ and $\rho = \sqrt{s\sigma}\, t$. Therefore, to find $\lambda_n$ one has to optimize the product

$$\exp\left(-\frac{t\sigma^2}{8D}\right)\exp(n\sqrt{s\sigma}\, t),$$

where the first factor is the probability of realizing a given value of $\sigma$ and the second factor is $\exp(n\rho)$. After optimizing the above expression over $\sigma$ one obtains $\exp(\lambda_n t)$ with

$$\lambda_n = \frac{3}{2^{5/3}} s^{2/3} D^{1/3} n^{4/3} \sim \lambda n^{4/3}. \tag{A.5}$$

The nonlinear dependence of $\lambda_n$ on $n$, $\lambda_n \propto n^{4/3}$, signals about strong intermittency of the flow. Note that in our anisotropic case the growth rates $\lambda_n$ increase with $n$ slower than in the isotropic case, where for the short-correlated flow $\lambda_n \propto n^2$ [15].

## References

[1] Monin A S and Yaglom A M 1975 *Statistical Fluid Mechanics* vols 1 and 2 (Cambridge, MA: MIT)
[2] Frisch U 1995 *Turbulence: The Legacy of A. N. Kolmogorov* (New York: Cambridge University Press)
[3] Vainshtein S and Zeldovich Ya 1972 Origin of magnetic fields in astrophysics *Sov. Phys.—Usp.* **15** 159
[4] Moffatt H K 1978 *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge University Press)
[5] Parker E N 1979 *Cosmic Magnetic Fields Their Origin and Activity* (Oxford: Clarendon)
[6] Zeldovich Ya B, Ruzmaikin A A and Sokolov D D 1990 *The Almighty Chance* (Singapore: World Scientific)
[7] Childress S and Gilbert A 1995 *Stretch Twist Fold: The Fast Dynamo* (Berlin: Springer)
[8] Rüediger G and Hollerbach R 2004 *The Magnetic Universe: Geophysical and Astrophysical Dynamo Theory* (New York: Wiley)
[9] Soward A M, Jones Ch A, Hughes D W and Weiss N O eds 2005 *Fluid Dynamics and Dynamos in Astrophysics and Geophysics* (Boca Raton, FL: CRC)
[10] Vishniac E T, Lazarian A and Cho J 2008 *Problems and Progress in Astrophysical Dynamos* (Berlin: Springer)
[11] Schekochihin A A, Cowley S C, Taylor S F, Maron L and McWilliams J C 2004 Simulations of the small-scale turbulent dynamo *Astrophys. J.* **612** 276–307
[12] Falkovich G, Gawedzki K and Vergassola M 2001 Particles fields in fluid turbulence *Rev. Mod. Phys.* **73** 913
[13] Kraichnan R and Nagarajan S 1967 Growth of turbulent magnetic fields *Phys. Fluids* **10** 859
[14] Kazantsev A P 1968 Enhancement of a magnetic field by a conducting fluid *Sov. Phys.—JETP* **26** 1031
[15] Chertkov M, Falkovich G, Kolokolov I and Vergassola M 1999 Small-scale turbulent dynamo *Phys. Rev. Lett.* **83** 4065–8
[16] Urpin V 2001 Kinematic turbulent dynamo in a shear flow *J. Geophysical and Astrophysical Fluid Dynamics* **95** 269–84
[17] Bird R B, Curtiss C F, Armstrong R C and Hassager O 1987 *Dynamics of Polymeric Liquids* 2nd edn (New York: Wiley)
[18] Doi M and Edwards S F 1988 *The Theory of Polymer Dynamics* (Oxford: Oxford University Press)
[19] Lumley J L 1972 On the solution of equations describing small scale deformation *Symp. Math.* **9** 315
[20] Balkovsky E, Fouxon A and Lebedev V 2000 Turbulent dynamics of polymer solutions *Phys. Rev. Lett.* **84** 4765–8

[21] Chertkov M, Kolokolov I, Lebedev V and Turitsyn K 2005 Polymer statistics in a random flow with mean shear *J. Fluid Mech.* **531** 251–60
[22] Groisman A and Steinberg V 2000 Elastic turbulence in a polymer solution flow *Nature* **405** 53
[23] Groisman A and Steinberg V 2001 Stretching of polymers in a random three-dimensional flow *Phys. Rev. Lett.* **86** 934
[24] Groisman A and Steinberg V 2001 Efficient mixing at low Reynolds numbers using polymer additives *Nature* **410** 905
[25] Landau L D and Lifshitz E M 1984 Course of theoretical physics *Electrodynamics of Continuous Media* (Oxford: Pergamon)
[26] Chertkov M, Kolokolov I and Vergassola M 1997 Inverse cascade and intermittency of passive scalar in one-dimensional smooth flow *Phys. Rev.* E **56** 5483–99
[27] Pope S B 1994 Lagrangian pdf methods for turbulent flows *Annu. Rev. Fluid Mech.* **26** 23
[28] Furstenberg H 1963 Noncommuting random products *Trans. Am. Math. Soc.* **108** 377
[29] Goldhirsch I, Sulem P-L and Orszag S A 1987 Stability and lyapunov stability of dynamical systems *Physica* D **27** 311
[30] Balkovsky E and Fouxon A 1999 Universal long-time properties of Lagrangian statistics in the batchelor regime and their application to the passive scalar problem *Phys. Rev.* E **60** 4164
[31] Ellis R 1985 *Entropy Large Deviations and Statistical Mechanics* (Berlin: Springer)
[32] Landau L D and Lifshitz E M 1980 Course of theoretical physics *Statistical Physics Part 1* (Oxford: Pergamon)
[33] Falkovich G, Kolokolov I, Lebedev V and Migdal A 1996 Instantons and intermittency *Phys. Rev.* E **54** 4896
[34] Zeldovich Ya B 1956 The magnetic field in the two-dimensional motion of a conducting turbulent fluid *Zh. Eksp. Teor. Fiz.* **31** 154
Zeldovich Ya B 1957 *Sov. Phys.—JETP* **4** 460
[35] Zeldovich Ya B and Ruzmaikin A A 1980 Magnetic field of a conducting fluid in two-dimensional motion *ZhETF* **78** 980
Zeldovich Ya B and Ruzmaikin A A 1980 *Sov. Phys.—JETP* **51** 493
[36] Zeldovich Ya B, Ruzmaikin A A, Molchanov S A and Sokolov D D 1984 Kinematic dynamo problem in a linear velocity field *J. Fluid Mech.* **144** 1
[37] Turitsyn K S 2007 Polymer dynamics in chaotic flows with a strong shear component *Zh. Eksp. Teor. Fiz.* **132** 746
Turitsyn K S 2007 *J. Exp. Theor. Phys.* **105** 655