# Dynamical correlations of two-dimensional vortex-like defects

V. V. Lebedev*)

*Landau Institute of Theoretical Physics, Russian Academy of Sciences, 117940 Moscow, Russia; Department of Physics of Complex Systems, Weizmann Institute of Science, 76100 Rehovot, Israel*

(Submitted 27 October 1999)
Pis'ma Zh. Éksp. Teor. Fiz. **70**, No. 10, 675–679 (25 November 1999)

High-order dynamical correlations of defects (quantum vortices, disclinations, etc.) in thin films are examined by starting from the Langevin equation for the defect motion. It is demonstrated that the dynamical correlation functions $F_{2n}$ of the vorticity or disclinicity behave as $F_{2n} \sim y^2/r^{4n}$, where $r$ is the characteristic scale and $y$ is the renormalized fugacity. Therefore below the Berezinskii–Kosterlitz–Thouless transition temperature the $F_{2n}$ are characterized by anomalous scaling exponents. The behavior differs strongly from the normal law $F_{2n} \sim F_2^n$ obeyed by equal-time correlation functions; the unequal-time correlation functions appear to be much larger. The phenomenon resembles intermittency in turbulence. © *1999 American Institute of Physics.* [S0021-3640(99)00822-1]

PACS numbers: 68.55.Ln

It is well known that defects such as quantum vortices, spin vortices, dislocations and disclinations play an essential role in physics of low-temperature phases of thin films. Berezinskii[1] and then Kosterlitz and Thouless[2] recognized that two-dimensional (2D) systems have a class of phase transitions related to defects. The main idea of their approach is that in 2D the defects can be treated as point objects interacting like charged particles. The low-temperature phase corresponds to a fluid consisting of bound uncharged defect–antidefect pairs, which is an insulator, whereas the high-temperature phase contains free charged particles and can be treated as a plasma. Correspondingly, in the low-temperature phase the correlation length is infinite, whereas in the high-temperature phase it is finite. An enormous number of papers have been devoted to different aspects of the problem (see, e.g., the surveys in Refs. 3–7). The scheme proposed by Kosterlitz and Thouless can be applied to superfluid and hexatic films and planar 2D magnets. It admits a generalization for crystalline films (see Refs. 8 and 9). There are also applications to superconductive materials, especially to high-$T_c$ superconductors (see, e.g., Ref. 10).

The dynamics of films in the presence of such defects was considered in Refs. 11 and 12. In those papers a complete set of equations was formulated describing both the motion of the defects and the hydrodynamic degrees of freedom. Then, to obtain macroscopic dynamical equations, an averaging over an intermediate scale was performed. In that procedure the ‘‘current density’’ related to the defects was replaced by an expression proportional to the average ‘‘electric field’’ and to gradients of the temperature and of the chemical potential. The resulting equations perfectly correspond to the problems solved in those papers.[11,12] Unfortunately, in that procedure information concerning high-order correlations of the defect motion is lost. That is the motivation for the present study, in which we shall examine these high-order correlations.

The static properties of the system of vortex-like defects in thin films can be described quite universally. The starting point of the description is the free energy $\mathcal{F}$ associated with the defects:

$$\mathcal{F}/T=-\sum_{i\neq j}\beta\, n_i n_j \ln\left(\frac{|x_i-x_j|}{a}\right)-\sum_j \ln y, \tag{1}$$

where the subscripts $i,j$ label defects, $x_i$ are the positions of the defects, $n_j=\pm 1$ are the ‘‘charges’’ of the defects, $a$ is a cutoff parameter of the order of the size of the defect core, and the coupling constant $\beta$ and the fugacity $y$ are dimensionless $T$-dependent factors. Expression (1) is correct for quantum vortices in superfluid films, for disclinations in hexatic films, and for spin vortices in 2D planar magnets. For dislocations in crystalline films, expression (1) has to be modified slightly.[8]

The presence of the pairs in the system leads to nontrivial ‘‘dielectric’’ properties of the medium. As a result, the interaction between the charges is modified; the effect can be described in terms of a scale-dependent coupling constant $\beta$ (Ref. 2). The dependence can be found in the framework of the scheme proposed by Kosterlitz.[13] Excluding pairs with separations from $a$ to $r$, we arrive at renormalized values of the parameters $\beta$ and $y$ which obey the following renormalization-group equations:

$$\frac{d\beta}{d\ln(r/a)}=-cy^2, \quad \frac{dy}{d\ln(r/a)}=(2-\beta)y, \tag{2}$$

where $c$ is a numerical factor of order unity. In the low-temperature phase, the coupling constant $\beta$ tends to a constant on large scales. The asymptotic value of $\beta$ is larger than 2; the critical value $\beta_c=2$ corresponds to the transition temperature. In the asymptotic region, where $\beta$ can be treated as $r$-independent, the renormalized fugacity $y$ remains $r$-dependent. Its asymptotic behavior can easily be extracted from Eq. (2): $y\propto r^{2-\beta}$. We see that in the low-temperature phase $y$ tends to zero as the scale increases. Thus the inequality $y\ll 1$ is satisfied for large scales in the low-temperature phase and probably in some region of scales above $T_c$.

We consider the correlation functions

$$F_{2n}(t_1,\dots,t_{2n};x_1,\dots,x_{2n})=\langle\rho(t_1,x_1)\dots\rho(t_{2n},x_{2n})\rangle \tag{3}$$

of the ‘‘charge density’’

$$\rho(x)=\sum_j n_j\delta(x-x_j). \tag{4}$$

For superfluid films the ‘‘charge density’’ (4) is proportional to the vorticity $\nabla\times\mathbf{v}_s$. In statics, one gets the estimate[14]

$$F_2(r)\sim y^2(r)/r^4, \tag{5}$$

where $r=|x_1-x_2|$. For high-order equal-time correlation functions the normal estimate $F_{2n}\sim F_2^n$ is valid under the condition $y\ll 1$.[15]

Following Ref. 11, we adopt the stochastic equation

$$\frac{dx_{j,\alpha}}{dt}=-\frac{D}{T}\left[\frac{\partial\mathscr{F}}{\partial x_{\alpha j}}+n_j\gamma\epsilon_{\alpha\beta}\frac{\partial\mathscr{F}}{\partial x_{\beta j}}\right]+\xi_{j,\alpha}, \tag{6}$$

which determines the trajectory of the $j$th vortex in a superfluid film. Here $\mathscr{F}$ is the free energy (1), $D$ is a diffusion coefficient, $\gamma$ is a dimensionless parameter, and $\xi_j$ are Langevin forces with the correlation function

$$\langle\xi_{i,\alpha}(t_1)\xi_{j,\beta}(t_2)\rangle=2D\delta_{ij}\delta_{\alpha\beta}\delta(t_1-t_2). \tag{7}$$

Equation (6) can be derived in the spirit of the procedure proposed by Hall and Vinen for a 3D superfluid (see Ref. 16). Huber[17] argued that the same equation is correct for spin vortices in planar 2D magnets. We believe that for $\gamma=0$, Eq. (6) is applicable to the dynamics of disclinations in hexatic films such as membranes, freely suspended films, and Langmuir films. We should also take into account annihilation and creation processes. They are characterized by a creation rate $\overline{R}(r)$, which is the probability density for a defect–antidefect pair with separation $r$ to be created per unit time per unit area, and by an annihilation rate $R(r)$, which is the probability for a defect–antidefect pair to annihilate per unit time if the pair is separated by a distance $r$. Actually $\overline{R}(r)$ and $R(r)$ are nonzero only if $r$ is of the order of the core size $a$.

To examine the correlation functions (3) we use the Doi technique,[18] which enables one to treat systems of classical particles in which creation and annihilation processes occur. The Doi technique is formulated in terms of creation $\hat{\psi}$ and annihilation $\psi$ operators which satisfy the same commutation rules as do those for Bose particles; we must introduce the fields $\psi_\pm$ and $\hat{\psi}_\pm$ corresponding to defects and antidefects, respectively. An effective Hamiltonian can be written in terms of the fields. Then one can formulate a conventional diagrammatic expansion in terms of $R$, $\overline{R}$, $\beta$, and the propagators of defects. Extracting blocks with small separations between the defects, one can pass to renormalized quantities. Details of the procedure will be published elsewhere.[19] Here we only present the results. The fugacity is expressed in terms of the renormalized values of the creation and annihilation rates as

$$\overline{R}(x)=\frac{y^2}{r^4}R(x). \tag{8}$$

The renormalized value of the annihilation rate is

$$R(x)=8\pi\beta\delta(x). \tag{9}$$

The renormalized values of $\beta$ and $y$ obey the same renormalization-group equations (2) as in statics. For $D$ and $\gamma$ we get the following renormalization-group equations:

$$\frac{dD}{d\ln(r/a)}\sim -y^2,\quad \frac{d\gamma}{d\ln(r/a)}\sim y^2, \tag{10}$$

which are analogous to the equation (2) for $\beta$. We conclude that the corrections to $D$ and $\gamma$ are small (and irrelevant) due to the small value of the fugacity.

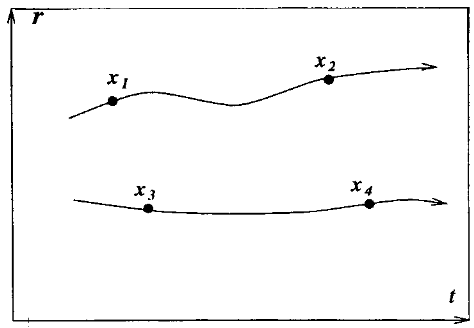


FIG. 1. Two trajectories passing through given points.

Next, we can examine the correlation functions (3), rewriting the charge density (4) as

$$\rho = \hat{\psi}_+ \psi_+ - \hat{\psi}_- \psi_- . \tag{11}$$

One should distinguish between contributions related to different numbers $k$ of defect–antidefect pairs passing through the points $x_m$ at given times $t_m$. They can be estimated as

$$F_{2n} \sim y^{2k}(r_*) r_*^{-4n} , \tag{12}$$

where we assume that all spatial separations are of the order of $r_*$ and all time intervals are of the order of $r_*^2/D$. We see that the ratio in (12) contains the $2k$th power of a dimensionless small parameter $y$. Thus we conclude that the leading contribution to $F_{2n}$ is due to a single defect–antidefect pair, which corresponds to $k=1$. The situation is illustrated in Fig. 1. Though the contribution associated with a number of defect–antidefect pairs contains an additional huge entropy factor, it has also an additional small factor associated with the small probability of observing defect–antidefect pairs with separations larger than the core radius; the smallness is due to the strong Coulomb attraction. In the large-scale limit when $\beta$ is saturated we have $F_{2n} \propto r_*^{-4(n-1)-2\beta}$. If some spatial separations among $|r_i - r_j|$ and/or some time intervals differ strongly, then one can formulate some simple rules. For example, if one of the time intervals $\tau$ is much larger than all values of $|r_i - r_j|^2/D$, then the correlation function behaves as $F_{2n} \propto \tau^{-\beta}$.

For the pair correlation function we have the same estimate (5) as in statics. The high-order correlation functions are much larger than their normal estimates in terms of the pair correlation function. Namely, in accordance with Eqs. (5) and (12) we have

$$F_{2n}/F_2^n \sim y^{-2n+2} \gg 1. \tag{13}$$

Let us explain in terms of our scheme the origin of the estimate $F_{2n} \sim F_2^n$ for the equal-time correlation functions. This estimate corresponds to $k=n$ in Eq. (12). The reason is that two defects cannot pass simultaneously through $2n$ points, and at least $k=n$ defect–antidefect pairs must be taken in order to obtain a nonzero contribution to the equal-time correlation function $F_{2n}$. The situation is illustrated in Fig. 2. Thus we have two different regimes: for equal-time and for unequal-time correlation functions. To establish the boundary between the regimes we should consider small time intervals in which the

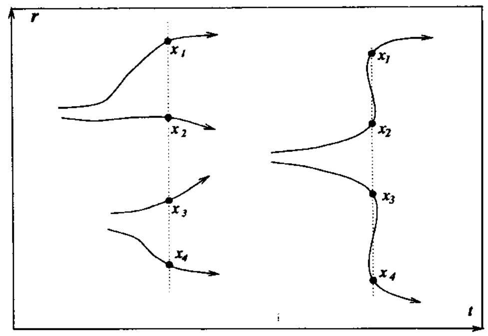


FIG. 2. Possible and impossible trajectories passing through four points at a given moment in time.

single-pair contribution is finite but small. The smallness is associated with diffusive ''smearing.'' The characteristic time over which the simultaneous (equal-time) regime passes into the nonsimultaneous (unequal-time) regime can be estimated from $Dt \sim r^2/|\ln[y(r)]|$, where $r$ is the characteristic spatial separation.

The effect considered resembles intermittency in turbulence (see, e.g., Ref. 20), which leads to large $r$-dependent factors in the ratios like (13) in the velocity correlation functions of a turbulent flow. However, in the case of defects the large $r$-dependent factors are related to the ultraviolet cutoff parameter, whereas for intermittency in turbulence the large $r$-dependent factors are related to the infrared (pumping) scale. Our situation is thus closer to the inverse cascade (see Ref. 21) realized on scales much larger than the pumping length. Our results can also be compared with nontrivial tails of probability distribution functions in the physics of disorder materials (see, e.g., Refs. 22 and 23).

I am grateful to G. Falkovich, K. Gawedzki, I. Kolokolov, and M. Vergassola for helpful discussions and to E. Balkovsky, A. Kashuba, and S. Korshunov for valuable remarks. This research was supported in part by a grant from the Israel Science Foundation, by a grant from the Minerva Foundation, and by the Landau–Weizmann Prize program.

*) e-mail: lebede@landau.ac.ru and lwlebede@wicc.weizmann.ac.il

[1] V. L. Berezinskii, Zh. Éksp. Teor. Fiz. **59**, 907 (1970) [Sov. Phys. JETP **32**, 493 (1971)]; Zh. Éksp. Teor. Fiz. **61**, 1144 (1971) [Sov. Phys. JETP **34**, 610 (1972)].
[2] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **5**, L124 (1972); **6**, 1181 (1973).
[3] J. M. Kosterlitz and D. J. Thouless, in *Progress in Low Temperature Physics*, Vol. VII B, edited by D. F. Brewer, North-Holland, Amsterdam (1978), p. 373.
[4] D. R. Nelson, in *Fundamental Problems in Statistical Mechanics*, Vol. V, edited by E. G. D. Cohen, North-Holland, New York (1980), p. 53.
[5] D. R. Nelson, in *Phase Transitions and Critical Phenomena*, Vol. 7, edited by C. Domb and J. L. Lebowitz, Academic Press, London (1983), p. 1.
[6] P. Minnhagen, Rev. Mod. Phys. **59**, 1001 (1987).
[7] Z. Gulacsi and M. Gulacsi, Adv. Phys. **47**, 1 (1998).
[8] D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977); B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. **41**, 121 (1978); B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. **41**, 519 (1978); A. P. Young, Phys. Rev. B **19**, 1855 (1979); D. R. Nelson and B. I. Halperin, Phys. Rev. B **19**, 2457 (1979).

[9] K. J. Strandburg, Rev. Mod. Phys. **60**, 161 (1988).
[10] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein *et al.*, Rev. Mod. Phys. **66**, 1125 (1994).
[11] A. Ambegaokar, B. I. Halperin, D. R. Nelson, and E. D. Siggia, Phys. Rev. Lett. **40**, 783 (1978); Phys. Rev. B **21**, 1806 (1980).
[12] A. Zippelius, B. I. Halperin, and D. R. Nelson, Phys. Rev. B **22**, 2514 (1980); R. Bruinsma, B. I. Halperin, and A. Zippelius, Phys. Rev. B **25**, 579 (1982).
[13] J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974).
[14] J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).
[15] L. P. Kadanoff, Ann. Phys. **120**, 39 (1979); L. P. Kadanoff and A. B. Zisook, Nucl. Phys. B **180**, 61 (1981).
[16] H. E. Hall and W. F. Vinen, Proc. R. Soc. London, Ser. A **238**, 215 (1956); W. F. Vinen, in Progress in *Low Temperature Physics*, Vol. III, North-Holland, Amsterdam (1961), p. 1; P. Nozieres and W. F. Vinen, Philos. Mag. **14**, 667 (1966).
[17] D. L. Huber, Phys. Rev. B **26**, 3758 (1982).
[18] M. Doi, J. Phys. A: Math. Gen. **9**, 1465, 1479 (1976).
[19] V. V. Lebedev, Phys. Rev. E, submitted.
[20] U. Frisch, *Turbulence: the Legacy of A. N. Kolmogorov*, Cambridge University Press, New York (1995).
[21] R. Kraichnan, Phys. Fluids **10**, 1417 (1967); R. Kraichnan, J. Fluid Mech. **47**, 525 (1971).
[22] I. Lifshits, S. Gredeskul, and A. Pastur, *Introduction to the theory of disordered systems*, Wiley Interscience, New York (1988).
[23] K. B. Efetov, *Supersymmetry in Disorder and Chaos*, Cambridge University Press (1997).

Published in English in the original Russian journal. Edited by Steve Torstveit.