Instanton for the Kraichnan passive scalar problem

E. Balkovsky[1] and V. Lebedev[1,2]
[1]Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
[2]Landau Institute for Theoretical Physics, Russian Academy of Sciences, Kosygina 2, Moscow 117940, Russia
(Received 2 June 1998)

We consider high-order correlation functions of the passive scalar in the Kraichnan model. Using the instanton formalism we find the scaling exponents ζ_n of the structure functions S_n for $n\gg1$ under the additional condition $d\zeta_2\gg1$ (where d is the dimensionality of space). At $n<n_c$ [where $n_c=d\zeta_2/2(2-\zeta_2)$] the exponents are $\zeta_n=(\zeta_2/4)(2n-n^2/n_c)$, while at $n>n_c$ they are n independent: $\zeta_n=\zeta_2 n_c/4$. We also estimate n-dependent factors in S_n, particularly their behavior at n close to n_c. [S1063-651X(98)04011-2]

PACS number(s): 47.27.Ak, 05.20.−y, 05.40.+j, 47.10.+g

INTRODUCTION

Anomalous scaling is probably the central problem of the theory of turbulence. In 1941 Kolmogorov formulated his famous theory of developed turbulence [1], where the scaling behavior of different correlation functions of the turbulent velocity was predicted. Experimentally one observes deviations from the scaling exponents, proposed by Kolmogorov [2–4]. It is recognized that the deviations are related to rare strong fluctuations making the main contribution into the correlation functions [5–7]. This phenomenon, which is usually called intermittency, is the most striking peculiarity of developed turbulence.

One of the classical objects in the theory of turbulence is a passive scalar advected by a fluid. The role of the passive scalar can be played by temperature or the density of pollutants. Correlation functions of the scalar in a turbulent flow possess a scaling behavior that was established by Obukhov [8] and Corrsin [9] in the frame of a theory analogous to that of Kolmogorov. Intermittency enforces deviations from the Obukhov-Corrsin exponents that appear to be even stronger than the deviations from the Kolmogorov exponents for the correlation functions of the velocity [10–13].

Unfortunately, a consistent theory of turbulence describing anomalous scaling has not been constructed yet. This accounts for the difficulties associated with the strong coupling inherent to developed turbulence. This is the reason for attempts to examine the intermittency phenomenon in the framework of different simplified models. The most popular model used for this purpose is Kraichnan's model of passive scalar advection [14], where the advecting velocity is believed to be short correlated in time and have a Gaussian distribution. That allows one to examine the statistics of the passive scalar in more detail.

The scalar in the Kraichnan model exhibits strong intermittency even if it is absent in the advecting velocity field. This was proved both theoretically [15–22] and numerically [23–25]. In the theoretical works the equation for the n-point correlation function F_n was solved assuming that different parameters, such as ζ_2, $2-\zeta_2$, or d^{-1}, are small (recall that ζ_2 is the exponent of the second-order correlation function of the passive scalar and d is the dimensionality of space). The order of the correlation functions that can be examined in the framework of the methods of the noted papers is bounded from above, which does not allow one to imagine the whole dependence of ζ_n on n. For that it would be enough to get the asymptotic behavior of ζ_n at $n\gg1$. There have been several attempts to find the scaling of the correlation functions for larger n. In the the work by Kraichnan [26] a closure was assumed enabling one to find ζ_n for any n. An alternative scheme was proposed in [27]. An attempt to solve the problem at large n was made in [28], where an n-independent asymptotic behavior was found.

In the present work we develop a technique based on the path-integral representation of the dynamical correlation functions of classical fields [29–31]. We use an idea, formulated in [32], that is related to the possibility of exploiting the saddle-point approximation in the path integral at large n. The saddle-point conditions are integro-differential equations describing an object that, in analogy to the quantum field theory, we call an instanton. The instantonic method was already successfully used in some contexts. Results concerning Burgers turbulence, conventional Navier-Stokes turbulence, and modifications of the Kraichnan model were obtained with the help of this method in Refs. [33–36]. The formalism presented in this paper enables one to find correlation functions of the passive scalar for arbitrary $n\gg1$ provided $d\zeta_2\gg1$.

The paper is organized as follows. In Sec. I we formulate the Kraichnan model, introduce notation, and write down the standard path integral representation for the correlation functions. This basic representation turns out to be unsuitable for the saddle-point approximation; therefore, we reformulate the problem in Sec. II. Passing to new variables that are Lagrangian separations, we get a path integral that already admits the use of the saddle-point approximation. In Sec. III we consider the instantonic equations for the case of the structure functions. We solve these equations in the limit $d\zeta_2\gg1$, which enables us to find the anomalous scaling and estimate the n dependence of S_n. The main results of the work are presented in Sec. III C and discussed in Conclusion. Details of calculations are given in Appendixes.

I. KRAICHNAN MODEL

Advection of a passive scalar θ by a velocity field $\boldsymbol{v}$ is described by the equation

$$\partial_t \theta + \boldsymbol{v}\boldsymbol{\nabla}\theta - \kappa \nabla^2 \theta = \phi, \tag{1.1}$$

where κ is the diffusion coefficient and ϕ is the source of the passive scalar (say, if θ corresponds to fluctuations of temperature, then ϕ represents the power of heaters). In a turbulent flow, $\boldsymbol{v}$ is a random function of time and space coordinates. The source ϕ is also assumed to be a random function. Then passive scalar correlation functions are determined by the statistics of $\boldsymbol{v}$ and ϕ. Usually, one is interested in simultaneous correlation functions $F_n = \langle \theta(\boldsymbol{r}_1) \cdots \theta(\boldsymbol{r}_n) \rangle$ since a large-scale velocity destroys temporal correlations in the Eulerian frame, whereas simultaneous objects are not influenced by it.

It is convenient to examine the anomalous scaling in terms of the structure functions

$$S_n(r) = \langle |\theta(\boldsymbol{r}/2) - \theta(-\boldsymbol{r}/2)|^n \rangle. \tag{1.2}$$

One expects a universal behavior of the structure functions in the convective interval of scales $r_d \ll r \ll L$, where r_d is the scale where the diffusivity becomes relevant and L is the correlation length of the scalar source ϕ. Namely, one observes a scaling dependence on r:

$$S_n(r) \propto r^{\zeta_n}. \tag{1.3}$$

In the frame of the Obukhov theory [8,9] $\zeta_n = (n/2)\zeta_2$. Therefore, the differences $(n/2)\zeta_2 - \zeta_n$, which are usually called anomalous exponents, characterize the anomalous scaling. One can write an estimate

$$S_n(r) \sim A_n [S_2(r)]^{n/2} \left(\frac{L}{r} \right)^{(n/2)\zeta_2 - \zeta_n}, \tag{1.4}$$

where A_n is an n-dependent factor. Note that Eq. (1.4) implies that the structure functions in the convective interval do not depend on the diffusion length r_d. The intermittency leads to the conclusion that values of the structure functions should be much larger than their naive Obukhov estimations [7]. Therefore, $(n/2)\zeta_2 - \zeta_n > 0$ and we conclude that these are the anomalous exponents that reflect the intermittency.

A. Formulation of the problem

In the Kraichnan model both $\boldsymbol{v}$ and ϕ are assumed to be independent random functions, δ correlated in time and described by Gaussian statistics homogeneous in space. Therefore, statistical properties of the fields are entirely characterized by the pair correlation functions

$$\langle \phi(t_1, \boldsymbol{r}_1) \phi(t_2, \boldsymbol{r}_2) \rangle = \chi(r_{12}) \delta(t_1 - t_2), \quad \chi(0) = P_2,$$

$$\langle v_\alpha(t_1, \boldsymbol{r}_1) v_\beta(t_2, \boldsymbol{r}_2) \rangle = \mathcal{V}_{\alpha\beta}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \delta(t_1 - t_2).$$

Here $\chi(r)$ is a smooth function decaying on the scale L, which is the pumping length. The constant P_2 has the meaning of the pumping rate of θ^2. The tensor $\mathcal{V}_{\alpha\beta}(r)$ has a characteristic scale L_v, which has the meaning of the pumping length of the velocity. We will assume that $L_v \gg L$. Since $r \ll L$ in the convective interval, we will need $\mathcal{V}_{\alpha\beta}$ only at $r \ll L_v$, where one can write

$$\mathcal{V}_{\alpha\beta}(\boldsymbol{r}) = \mathcal{V}_0 \delta_{\alpha\beta} - \mathcal{K}_{\alpha\beta}(\boldsymbol{r}). \tag{1.5}$$

The quantity $\mathcal{V}_0$ is an r-independent constant that is the main contribution to the velocity correlation function on scales less than the velocity pumping length L_v. Nevertheless, besides $\mathcal{V}_0$, we should also keep a small r-dependent correction $\mathcal{K}$ since $\mathcal{V}_0$ corresponds to advection homogeneous in space and therefore does not contribute to simultaneous correlation functions of θ.

The velocity correlation function is assumed to possess some scaling properties, namely, $\mathcal{K}(\boldsymbol{r}) \propto r^{2-\gamma}$, where the exponent γ characterizes the roughening degree of the velocity field. The field is smooth in space at $\gamma = 0$ and is extremely irregular at $\gamma = 2$. We will treat an arbitrary γ satisfying the inequality $0 < \gamma < 2$. The tensorial structure of $\mathcal{K}_{\alpha\beta}$ is determined by the incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$, implied in the Kraichnan model

$$\mathcal{K}_{\alpha\beta}(\boldsymbol{r}) = \frac{D}{d} r^{-\gamma} \left[\frac{2-\gamma}{d-1} (r^2 \delta_{\alpha\beta} - r_\alpha r_\beta) + r^2 \delta_{\alpha\beta} \right]. \tag{1.6}$$

Here d is the dimensionality of space and D is a constant characterizing the strength of velocity fluctuations. One assumes that the fluctuations are strong enough to ensure the large value of the Péclet number, that is,

$$DL^{2-\gamma} \gg \kappa. \tag{1.7}$$

The inequality (1.7) ensures the existence of the convective interval of scales since it can be rewritten as $r_d \ll L$, where r_d is the diffusive length

$$r_d^{2-\gamma} \sim \kappa/D. \tag{1.8}$$

The assumption of the Gaussian nature and zero correlation time for the fields $\boldsymbol{v}$ and ϕ allows one to derive a closed partial differential equation for the nth order correlation function F_n of θ [14,37,17]. For the simultaneous pair correlation function $F_2(r_{12}) = \langle \theta(t, \boldsymbol{r}_1) \theta(t, \boldsymbol{r}_2) \rangle$ one can solve the equation and find the explicit expression for F_2. In the convective interval [14]

$$S_2(r) = 2[F_2(0) - F_2(r)] \sim \frac{P_2}{D} r^\gamma. \tag{1.9}$$

Comparing Eq. (1.9) with Eq. (1.3), one concludes that the exponent γ introduced by Eq. (1.6) directly determines the scaling of the second-order structure function $\zeta_2 = \gamma$.

However, for $n > 2$ the equations for F_n are too complicated to be integrated exactly. In [16–19] the equations were analyzed in the limits $2 - \gamma \ll 1$ and $d\gamma \gg 1$, where the statistics of the passive scalar is close to Gaussian. The analysis led to an anomalous scaling that can be expressed in terms of the exponents ζ_n of the structure functions (1.2) and (1.3),

$$\zeta_n = \frac{n\gamma}{2} - \frac{2-\gamma}{2(d+2)} n(n-2). \tag{1.10}$$

This expression covers both limit cases $2 - \gamma \ll 1$ and $d\gamma \gg 1$. The first term on the right-hand side of Eq. (1.10) represents the normal scaling, whereas the second one is just the anomalous scaling exponent. The calculations leading to Eq. (1.10) are correct if the anomalous contribution is much smaller than the normal one, which implies the inequality

$$n \ll \frac{d\gamma}{2-\gamma}. \tag{1.11}$$

Below we will develop a different approach to the problem. It will allow us to find the exponents ζ_n [Eq. (1.3)] of the structure correlation functions (1.2) for any order $n \gg 1$ under the same additional condition $d\gamma \gg 1$ as in [17,18].

B. Path integral

Generally, the statistics of classical fields in the presence of random forces can be examined with the help of the field technique formulated in [29–31]. In the framework of the technique, correlation functions are calculated as path integrals with the weight $\exp(i\mathcal{I})$, where $\mathcal{I}$ is the effective action related to dynamical equations for the fields. For the passive scalar in the Kraichnan model the effective action is

$$i\mathcal{I}_\theta = i\int dt\, d\boldsymbol{r}[p\partial_t\theta + p\boldsymbol{v}\nabla\theta + \kappa\nabla p\nabla\theta] - \frac{1}{2}\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2 \chi(|\boldsymbol{r}_1-\boldsymbol{r}_2|)p(t,\boldsymbol{r}_1)p(t,\boldsymbol{r}_2), \tag{1.12}$$

where p is an auxiliary field conjugated to θ. The first term in the effective action (1.12) is directly related to the left-hand side of Eq. (1.1). The quadratic in the p term in Eq. (1.12) appears as a result of averaging over the statistics of the pumping ϕ.

Simultaneous correlation functions of θ can be represented as functional derivatives of the generating functional

$$\mathcal{Z}(\lambda) = \left\langle \exp\left[i\int d\boldsymbol{r}\,\lambda(\boldsymbol{r})\theta(t=0,\boldsymbol{r})\right]\right\rangle, \tag{1.13}$$

where angular brackets designate averaging over the statistics of ϕ and $\boldsymbol{v}$. With the help of the action (1.12) the generating functional can be rewritten as the path integral

$$\mathcal{Z}(\lambda) = \int \mathcal{D}\theta\,\mathcal{D}p\,\mathcal{D}\boldsymbol{v}\,\exp\left[-\mathcal{F}(\boldsymbol{v}) + i\mathcal{I}_\theta + i\int d\boldsymbol{r}\,\lambda(\boldsymbol{r})\theta(t=0,\boldsymbol{r})\right]. \tag{1.14}$$

Here $\mathcal{F}(\boldsymbol{v})$ determines the statistics of the velocity field. Since we assume the Gaussian nature of the statistics, $\mathcal{F}(\boldsymbol{v})$ is a functional of second order over $\boldsymbol{v}$ with the kernel determined by the pair correlation function (1.5). Knowing $\mathcal{Z}(\lambda)$, one can restore the probability distribution function (PDF) of θ. It is convenient to treat the PDF of a particular object

$$\vartheta = \int d\boldsymbol{r}\,\beta(\boldsymbol{r})\theta(t=0,\boldsymbol{r}), \tag{1.15}$$

with a given function $\beta(\boldsymbol{r})$. For example, the set of the structure functions (1.2) can be assembled into the PDF of the scalar difference in two points $\theta(\boldsymbol{r}/2)-\theta(-\boldsymbol{r}/2)$, which is the object (1.15) with $\beta(\boldsymbol{r}_1)=\delta(\boldsymbol{r}_1-\boldsymbol{r}/2)-\delta(\boldsymbol{r}_1+\boldsymbol{r}/2)$. The PDF of ϑ is written as

$$\mathcal{P}(\vartheta) = \int_{-\infty}^{\infty}\frac{dy}{2\pi}\exp(-iy\vartheta)\mathcal{Z}[y\beta(\boldsymbol{r})]. \tag{1.16}$$

Moments of ϑ are then expressed as

$$\langle|\vartheta|^n\rangle = \int_{-\infty}^{\infty} d\vartheta\,|\vartheta|^n\mathcal{P}(\vartheta). \tag{1.17}$$

We will be interested in the high-order correlation functions of ϑ or, in other words, we consider the limit $n \gg 1$. This is equivalent to examining the large ϑ tail of the PDF (1.16). One could expect [32] that the tail can be calculated in the saddle-point approximation since there is a large parameter ϑ in the corresponding path integral. Unfortunately, direct application of the method to the integral (1.16) or to the moments (1.17) does not lead to success.

To recognize the reason, let us consider the transformation of the variables [32] (see also [35])

$$\boldsymbol{v}\to X\boldsymbol{v}, \quad p\to Xp, \quad t\to X^{-1}t, \quad y\to Xy, \quad \kappa\to X\kappa.$$

One can check that under this transformation all the terms in the square brackets on the right-hand side of Eq. (1.14) acquire the factor X, which means that in the saddle-point approximation $\ln[\mathcal{Z}(y\beta)] = yf(y/\kappa)$ with some unknown function f. On the other hand, we expect that correlation functions of the scalar itself (but not of its gradient, for example) do not depend on the diffusivity and the results of the works [15–19,21,22,37] confirm the expectation. Then, at small κ the function f remains a κ-independent constant and we obtain

$$\ln\mathcal{Z}(y\beta) \propto |y|. \tag{1.18}$$

Unfortunately, Eq. (1.18) does not help to restore $\mathcal{P}(\vartheta)$ since after substituting it into Eq. (1.16) we realize that the characteristic value of y in the integral can be estimated as $y \sim \vartheta^{-1}$. Therefore, at large ϑ the main contribution to the integral is determined by the region where Eq. (1.18) does not work.

We conclude that the naive instantonic approach to the problem fails. The reason is that for the instanton the velocity field is fixed (does not fluctuate) in time and space. Obviously, a saddle-point solution is anisotropic because of the incompressibility condition $\mathrm{div}\,\boldsymbol{v}=0$. Fluctuations related to smooth variations of the anisotropy axis in time and space are strong and destroy the saddle-point approximation for the tail of the PDF $\mathcal{P}(\vartheta)$ or for the high moments of ϑ. Thus we should transform the problem to more adequate variables, where fluctuations of the velocity are partly taken into account. This is the only chance to construct an instanton with weak fluctuations on its background. This is the goal of the next section.

II. LAGRANGE FORMULATION

As we mentioned above, the diffusivity κ does not enter the result for the structure functions. Therefore, we will assume $\kappa=0$ in all the following calculations. However, one should be careful since in this case it is impossible to deal with point objects. To provide a regularization, we should assume that the characteristic scales of the function β in Eq.

(1.15) are larger than r_d. In addition, the scales are to be much smaller than L since we are going to examine correlation functions in the convective interval.

In the diffusionless case the left-hand side of Eq. (1.1) describes the field θ moving together with the fluid. Then it is natural to pass into the Lagrangian frame where the process is trivial. For that purpose we introduce Lagrangian trajectories $\varrho(t)$ that obey the equation

$$\partial_t \varrho = \boldsymbol{v}(t,\varrho). \tag{2.1}$$

We will label the trajectories by the positions of fluid particles at $t=0$: $\varrho(t=0)=\boldsymbol{r}$. Equation (1.1) (where κ is omitted) can easily be solved in terms of the Lagrangian trajectories

$$\theta(0,\boldsymbol{r}) = \int_{-\infty}^{0} dt\, \phi[t,\varrho(t,\boldsymbol{r})], \tag{2.2}$$

Since we are interested in the field θ at $t=0$, due to causality, the integration is performed over negative time. Therefore, Eq. (2.1) should be solved backward in time.

A simultaneous nth-order correlation function of θ can be written as the product of n integrals (2.2), averaged over the statistics of $\boldsymbol{v}$ and ϕ. In this representation, averaging over the pumping is very simple. For example, the two-point correlation function is

$$F_2 = \int_{-\infty}^{0} dt \langle \chi(R_{12})\rangle_v , \tag{2.3}$$

$$R_{12}(t) \equiv R(t,\boldsymbol{r}_1,\boldsymbol{r}_2) = |\varrho(t,\boldsymbol{r}_1) - \varrho(t,\boldsymbol{r}_2)|. \tag{2.4}$$

The angular brackets $\langle\,\rangle_v$ in Eq. (2.3) denote averaging over the statistics of $\boldsymbol{v}$ only since the statistics of ϕ is already accounted for there. Similar formulas can be written for correlation functions of higher orders. Once this is done, one can assemble them into the generating functional (1.13)

$$\mathcal{Z}(\lambda) = \left\langle \exp\left\{ -\frac{1}{2}\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2\, \chi(R_{12})\lambda_1\lambda_2 \right\} \right\rangle_v , \tag{2.5}$$

where $\lambda_{1,2}=\lambda(\boldsymbol{r}_{1,2})$. Calculating the moments of the object (1.15) in accordance with Eqs. (1.16) and (1.17) we get

$$\langle |\vartheta|^n \rangle = \int \frac{dy\, d\vartheta}{2\pi} \langle \exp(-\mathcal{F}_\lambda - iy\vartheta + n \ln|\vartheta|)\rangle_v , \tag{2.6}$$

$$\mathcal{F}_\lambda = \frac{y^2}{2}\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2\, \chi(R_{12})\beta(\boldsymbol{r}_1)\beta(\boldsymbol{r}_2). \tag{2.7}$$

At this point, we would like to stress the close connection between the statistics of the passive scalar and that of Lagrangian trajectories [40], which can be seen from Eq. (2.5).

A. Statistics of Lagrangian separations

Equations (2.5) and (2.6) show that the correlation functions we are interested in are expressed via the average of $\exp(-\mathcal{F}_\lambda)$ over the velocity. Note that $\mathcal{F}_\lambda$ [Eq. (2.7)] depends only on the absolute values $R_{12}(t)$ of Lagrangian differences (2.4). Therefore, instead of averaging over the statistics of $\boldsymbol{v}$, one could find the answer by averaging over the statistics of the Lagrangian separations R_{12}. Due to zero correlation time of the velocity field, the statistical properties of the field R_{12} appear to be relatively simple.

To establish the statistics of R_{12} we start from the relation

$$\gamma^{-1}\partial_t R_{12}^{\gamma} = \zeta_{12} \equiv R_{12}^{\gamma-2} R_{12\alpha}(v_{1\alpha} - v_{2\alpha}), \tag{2.8}$$

following from Eqs. (2.1) and (2.4). As shown in Appendix , the average value of ζ_{12} is nonzero:

$$\langle \zeta_{12}\rangle = -D. \tag{2.9}$$

Next, exploiting the expression (1.5) for the velocity correlation function, one can find the irreducible pair correlation function

$$\langle\langle \zeta_{12}(t_1)\zeta_{34}(t_2)\rangle\rangle = \frac{2D}{d} Q_{12,34}\delta(t_1 - t_2). \tag{2.10}$$

The explicit expression for the function Q is rather cumbersome:

$$\begin{aligned} Q_{12,34} = {} & \frac{d+1-\gamma}{4(d-1)} R_{12}^{\gamma-2} R_{34}^{\gamma-2} (R_{23}^{2-\gamma} + R_{14}^{2-\gamma} - R_{13}^{2-\gamma} - R_{24}^{2-\gamma}) \\ & \times (R_{23}^2 + R_{14}^2 - R_{13}^2 - R_{24}^2) - \frac{2-\gamma}{8(d-1)} R_{12}^{\gamma-2} R_{34}^{\gamma-2} \\ & \times \left\{ \frac{1}{R_{13}^{\gamma}} (R_{12}^2 + R_{13}^2 - R_{23}^2)(R_{13}^2 + R_{34}^2 - R_{14}^2) \right. \\ & + \frac{1}{R_{23}^{\gamma}} (R_{12}^2 + R_{23}^2 - R_{13}^2)(R_{34}^2 + R_{23}^2 - R_{24}^2) \\ & + \frac{1}{R_{14}^{\gamma}} (R_{12}^2 + R_{14}^2 - R_{24}^2)(R_{14}^2 + R_{34}^2 - R_{13}^2) \\ & \left. + \frac{1}{R_{24}^{\gamma}} (R_{12}^2 + R_{24}^2 - R_{14}^2)(R_{34}^2 + R_{24}^2 - R_{23}^2) \right\}. \end{aligned} \tag{2.11}$$

It can be found from the definition of ζ_{12} [Eq. (2.8)], formula (1.6), and relations such as

$$\boldsymbol{R}_{12}\cdot \boldsymbol{R}_{13} = \frac{1}{2}(R_{12}^2 + R_{13}^2 - R_{23}^2),$$

$$\boldsymbol{R}_{12}\cdot \boldsymbol{R}_{34} = \frac{1}{2}(R_{14}^2 + R_{23}^2 - R_{13}^2 - R_{24}^2).$$

In the spirit of the conventional procedure [29–31], one can assert that any average over the statistics of R_{12} can be found as the path integral over R_{12} and over an auxiliary field $m_{12} \equiv m(t,\boldsymbol{r}_1,\boldsymbol{r}_2)$ with the weight

$$\left\langle \exp\left[i\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2 (m_{12}\gamma^{-1}\partial_t R_{12}^{\gamma} - m_{12}\zeta_{12}) \right] \right\rangle_v ,$$

where angular brackets mean averaging over the statistics of the velocity. Since ζ_{12} is δ correlated in time, the average can be expressed in terms of Eqs. (2.9) and (2.10) only. The result is $\exp(i\mathcal{I}_R)$, where

$$i\mathcal{I}_R = i\int_{-\infty}^{0} dt \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 m_{12}(\gamma^{-1}\partial_t R_{12}^{\gamma} + D) - \frac{D}{d}\int_{-\infty}^{0} dt \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_3 d\boldsymbol{r}_4\, Q_{12,34} m_{12} m_{34}. \tag{2.12}$$

Now, instead of Eq. (2.6) we can write

$$\langle|\vartheta|^n\rangle = \int \frac{dy\, d\vartheta}{2\pi}\int \mathcal{D}R\,\mathcal{D}m\, e^{i\mathcal{I}_R - \mathcal{F}_\lambda - iy\vartheta + n\ln|\vartheta|}. \tag{2.13}$$

The integration in Eq. (2.13) is performed over functions of t, $\boldsymbol{r}_1$, and $\boldsymbol{r}_2$ with some boundary conditions imposed on them. The condition for the field R_{12} follows from $\varrho(0) = \boldsymbol{r}$ and reads

$$R_{12}(t=0) = |\boldsymbol{r}_1 - \boldsymbol{r}_2|. \tag{2.14}$$

The boundary condition for the field m_{12} should be $m_{12}(-\infty) = 0$ since we deal with free integration over R_{12} in the remote past. Note that due to the definition (2.4), the triangular inequalities

$$R_{12} + R_{23} > R_{13}, \tag{2.15}$$

should be satisfied for any three points. Actually, the inequalities are constraints that should be imposed on the field R_{12} when integrating in Eq. (2.13).

B. General instantonic equations

In the preceding subsection we derive a formula (2.13) for $\langle|\vartheta|^n\rangle$. Its calculation is equivalent to solving some nonlinear field theory. It looks infeasible to perform this task. We are going to calculate the integral (2.13) in the saddle-point approximation regarding the number n large enough. To be consistent, when doing the procedure one should remember about the constraints (2.15). Unfortunately, it is very hard to take them into account explicitly. We will ignore the constraints, which is correct under the following conditions. First, the inequalities (2.15) should be valid in the instantonic solution. Second, fluctuations on the background of the instanton should be weak (this is also the applicability condition of the instantonic formalism itself). We argue in Appendixes B 1 and B 2 that those conditions are satisfied if

$$d\gamma \gg 1. \tag{2.16}$$

Note also that for the condition (2.16) fluctuations of a Lagrangian separation near its average value are weak (see Appendix A 2). The inequality (2.16) will be implied below.

Thus we obtain from the integral (2.13) in the saddle-point approximation

$$\langle|\vartheta|^n\rangle \sim \exp(i\mathcal{I}_R - \mathcal{F}_\lambda - iy\vartheta + n\ln|\vartheta|)|_{\text{inst}}. \tag{2.17}$$

Here solutions of the instantonic equations should be substituted, which are extremum conditions for the argument of the exponent on the right-hand side of Eq. (2.13). Variation over m_{12} and R_{12} gives the following instantonic equations

$$i(\gamma^{-1}\partial_t R_{12}^{\gamma} + D) = 2\frac{D}{d}\int d\boldsymbol{r}_3 d\boldsymbol{r}_4\, Q_{12,34} m_{34}, \tag{2.18}$$

$$iR_{12}^{\gamma-1}\partial_t m_{12} + \frac{D}{d}\int d\boldsymbol{r}_3 d\boldsymbol{r}_4 \left\{ 2\frac{\partial Q_{12,34}}{\partial R_{12}} m_{12} m_{34} + 4\frac{\partial Q_{13,24}}{\partial R_{12}} m_{13} m_{24} \right\} = -\frac{y^2}{2}\chi'(R_{12})\beta(\boldsymbol{r}_1)\beta(\boldsymbol{r}_2). \tag{2.19}$$

The extremum conditions over y and ϑ read

$$\vartheta = iy\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2\, \chi(R_{12})\beta(\boldsymbol{r}_1)\beta(\boldsymbol{r}_2), \tag{2.20}$$

$$iy = n/\vartheta. \tag{2.21}$$

Note that only Eqs. (2.18) and (2.19) are true dynamical equations, carrying the information about the dynamics of the flow, whereas Eqs. (2.20) and (2.21) are constraints imposed on the instantonic solution. One needs to add to Eqs. (2.18) and (2.19) some boundary conditions. The value of the field R_{12} is fixed at $t=0$ by Eq. (2.14). As for the field m_{12}, we already noted that it should tend to zero when $t \to -\infty$. It can be understood as the extremum condition that appears after variation of the effective action over the boundary value of R_{12} in the remote past.

One can easily establish the asymptotic behavior of R_{12} at $|t| \to \infty$. There the field R_{12} grows and loses its dependence on $\boldsymbol{r}_{1,2}$. The field m_{12} tends to its ''vacuum'' zero value at $|t| \to \infty$. Therefore, at large $|t|$ the term with m_{12} in Eq. (2.18) can be omitted and we find

$$R^{\gamma} \approx \gamma D|t|. \tag{2.22}$$

The expression (2.22) is nothing but the Richardson law for divergence of Lagrangian trajectories [38]. Let us stress that now it holds on the classical (mean-field) level, without taking into account fluctuations on the background of the instanton. To clarify this point, notice that if the velocity field is a deterministic function of time and space (as it is for the naive instanton discussed above), then the Richardson law cannot be valid for all the Lagrangian trajectories. In our instanton we get rid of the velocity field that resulted in the emergence of the Richardson law. Note that the triangle inequalities (2.15) are obviously satisfied both for (2.14) and for the asymptotic behavior (2.22).

The expression for the action appearing in Eq. (2.13) is

$$i\mathcal{I} \equiv i\mathcal{I}_R - \mathcal{F}_\lambda = i\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2\, \gamma^{-1} m_{12}\partial_t R_{12}^{\gamma} - E, \tag{2.23}$$

$$E=\frac{y^2}{2}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \chi(R_{12})\beta(\boldsymbol{r}_1)\beta(\boldsymbol{r}_2)-iD\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 m_{12}+\frac{D}{d}\int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 d\boldsymbol{r}_3 d\boldsymbol{r}_4 Q_{12,34} m_{12} m_{34}. \tag{2.24}$$

We see from Eq. (2.23) that the quantity E plays the role of the Hamiltonian function of the system, while Eqs. (2.18) and (2.19) are canonical equations corresponding to the Hamiltonian function. Since E does not explicitly depend on time t, its value (which can be called energy) is conserved. Actually, the energy is zero on the instantonic solution since at $t\to-\infty$ we have $m_{12}\to 0$ and $R_{12}\to\infty$. Note that since the Hamiltonian (2.24) explicitly depends on the coordinates via β, there is no ''momentum'' conservation law.

Before proceeding to the solution of the instanton equations, let us make a remark concerning fluctuations on the background of the instanton. In the linear approximation over the fluctuations we obtain an estimate for the typical fluctuation of R^γ,

$$(\delta R^\gamma)^2\sim \gamma D R^\gamma |t| d^{-1}. \tag{2.25}$$

Note that the fluctuations of R tend to zero when $t\to 0$ since R_{12} is fixed at $t=0$. Comparing the estimate (2.25) with Eq. (2.22), we obtain

$$(\delta R^\gamma)^2/R^{2\gamma}\sim d^{-1}. \tag{2.26}$$

We conclude that the fluctuations on the background of our instanton are weak provided $d\gg 1$. The above evaluations are rough and need a more accurate analysis (see Appendix B 2). Nevertheless, they show that the Richardson behavior (2.22) inherent for our instanton suppresses fluctuations on its background.

The system (2.18) and (2.19) consists of two nonlinear integro-differential equations with boundary conditions imposed on the opposite sides of the time interval, that is, at $t=0$ for R_{12} and at $t=-\infty$ for m_{12}. Therefore, in the general case it is very difficult to solve the instanton equations. Nevertheless, one can hope that for some particular objects the system of equations can be reduced to a simpler form allowing the complete solution. This hope comes true for the structure functions.

III. INSTANTON FOR STRUCTURE FUNCTIONS

Using the general scheme developed in Sec. II, we will examine the expressions for the structure functions (1.2) at large n. In other words, we will be interested in the statistics of the passive scalar difference taken at the points separated by the distance $\boldsymbol{r}$. Since the diffusivity is neglected, we cannot examine the difference $\theta(\boldsymbol{r}/2)-\theta(-\boldsymbol{r}/2)$ itself. Nevertheless, we can treat the statistics of the differences averaged over separations near $\boldsymbol{r}$. So we should consider the object (1.15) with

$$\beta(\boldsymbol{r}_1)=\delta_\Lambda\left(\boldsymbol{r}_1-\frac{\boldsymbol{r}}{2}\right)-\delta_\Lambda\left(\boldsymbol{r}_1+\frac{\boldsymbol{r}}{2}\right). \tag{3.1}$$

Here $\delta_\Lambda(\boldsymbol{r})$ is the function with the width $\Lambda^{-1}\gg r_d$ satisfying the condition $\int d\boldsymbol{r}\,\delta_\Lambda(\boldsymbol{r})=1$, which can be called a smeared δ function. Then we can write

$$S_n\approx\langle|\vartheta|^n\rangle\sim\exp(i\mathcal{I}-n+n\ln|\vartheta|)|_{\text{inst}}, \tag{3.2}$$

where we used Eq. (2.17) and substituted Eq. (2.21).

A. Reduction

Now we turn to the instantonic equations (2.18) and (2.19). Let us observe that since the source on the right-hand side of Eq. (2.19) is proportional to $\beta(\boldsymbol{r}_1)\beta(\boldsymbol{r}_2)$, the field m_{12} can be approximated as

$$m_{12}=\frac{im_+}{2}\left\{\delta_\Lambda\left(\boldsymbol{r}_1-\frac{\boldsymbol{r}}{2}\right)\delta_\Lambda\left(\boldsymbol{r}_2-\frac{\boldsymbol{r}}{2}\right)+\delta_\Lambda\left(\boldsymbol{r}_1+\frac{\boldsymbol{r}}{2}\right)\delta_\Lambda\left(\boldsymbol{r}_2+\frac{\boldsymbol{r}}{2}\right)\right\}-\frac{im_-}{2}\left\{\delta_\Lambda\left(\boldsymbol{r}_1-\frac{\boldsymbol{r}}{2}\right)\times\delta_\Lambda\left(\boldsymbol{r}_2+\frac{\boldsymbol{r}}{2}\right)+\delta_\Lambda\left(\boldsymbol{r}_1+\frac{\boldsymbol{r}}{2}\right)\delta_\Lambda\left(\boldsymbol{r}_2-\frac{\boldsymbol{r}}{2}\right)\right\}, \tag{3.3}$$

where $m_\pm$ are functions of time only. Writing it, we implicitly assumed that the field R_{12} is smooth near the points $\pm\boldsymbol{r}/2$. Then the relations (2.20) and (2.21) give

$$\vartheta^2=2n\int_{-\infty}^0 dt\{\chi(R_+)-\chi(R_-)\}, \tag{3.4}$$

where we introduced

$$R_+(t)=R(t,\boldsymbol{r}/2,\boldsymbol{r}/2),\quad R_-(t)=R(t,\boldsymbol{r}/2,-\boldsymbol{r}/2). \tag{3.5}$$

Substituting the expression (3.3) into the Eqs. (2.18) and (2.19), we obtain a closed system of ordinary differential equations for $m_\pm$ and $R_\pm$. It is convenient to proceed in terms of the effective action. Substituting Eq. (3.3) into Eq. (2.23), we get

$$i\mathcal{I}=\int_{-\infty}^0 dt[\gamma^{-1}(m_-\partial_t R_-^\gamma-m_+\partial_t R_+^\gamma)-E], \tag{3.6}$$

$$E=y^2\{\chi(R_+)-\chi(R_-)\}+D(m_+-m_-)-\frac{D(2-\gamma)}{4d(d-1)}\{m_-^2\varphi_1+2m_-m_+\varphi_2+m_+^2\varphi_3\}. \tag{3.7}$$

Here we introduced the designations

$$\varphi_1=\frac{4(d+1-\gamma)}{2-\gamma}R_-^{2\gamma-4}[R_-^{2-\gamma}-R_+^{2-\gamma}][R_-^2-R_+^2]-R_-^{2\gamma-4}\left[R_+^{4-\gamma}+\frac{(2R_-^2-R_+^2)^2}{R_-^\gamma}\right], \tag{3.8}$$

$$\varphi_2=R_+^\gamma\left[\frac{R_+^{2-\gamma}}{R_-^{2-\gamma}}+\frac{2R_-^2-R_+^2}{R_-^2}\right],\quad \varphi_3=-R_+^\gamma\left[1+\frac{R_+^\gamma}{R_-^\gamma}\right].$$

Since the effective action (3.6) depends only on the functions $m_\pm(t)$ and $R_\pm(t)$, one can obtain the system of ordinary differential equations for the functions as extremum conditions of the action. The boundary conditions for the equations are $R_+=0$ and $R_-=r$ at $t=0$ [see Eq. (2.14)] and $m_\pm\to 0$ at $t\to-\infty$. Resolution of the system allows one to find $m_\pm$ and $R_\pm$ as functions of time. Once they are known, it is possible to restore the function R_{12} in the whole space from Eq. (2.18). The problem is discussed in Appendix B 1. There we argue that the function R_{12} is really smooth in space, which is a justification of the procedure described.

Since we accept Eq. (2.16) $d\gg 1$. Using the inequality, one can keep in the functions (3.8) only the terms of the main order over d. This means that one can neglect in Eq. (3.8) the second contribution to φ_1 in comparison to the first one and also φ_2,φ_3 in comparison to φ_1. Potentially this procedure is dangerous. We will show that due to the smallness of r/L, the intervals where $R_- - R_+ \ll R_-$ play an important role. Then we see that it is the difference of $R_\pm$ that enters the first term in φ_1, while the others do not contain this smallness. Therefore, we observe cancellations that could lead to a competition of d and L/r (the latter parameter is considered as the largest in the problem). To check the possibility, we performed calculations keeping all the terms in Eq. (3.8). The calculations are sketched in Appendix C. They show that in the final expressions only combinations of $\varphi_{1,2,3}$ containing the same cancellations are of importance. The legitimacy of the procedure is proved.

Omitting $\varphi_{2,3}$ in the expression (3.7) and then varying the action (3.6) over m_+, we get a trivial equation for R_+,

$$\gamma^{-1}\partial_t R_+^\gamma = -D. \tag{3.9}$$

Its solution, satisfying the boundary condition $R_+(0)=0$, is simply

$$R_+^\gamma = \gamma D|t|. \tag{3.10}$$

To examine the behavior of R_- it is convenient to pass to the new variables

$$R_+ = Le^{\xi}, \quad R_-^\gamma = R_+^\gamma(1+v), \quad \mu = m_- R_+^\gamma. \tag{3.11}$$

As time t goes from 0 to $-\infty$, the variable ξ runs from $-\infty$ to $+\infty$ and v runs from $+\infty$ to 0. The latter is clear from the asymptotic behavior $R_-^\gamma \approx R_+^\gamma = \gamma D|t|$ at $t\to-\infty$. The relation (3.4) in terms of the new variables is

$$\vartheta^2 = 2n\frac{L^\gamma}{D}\int_{-\infty}^{+\infty} d\xi\, e^{\gamma\xi}[\chi(R_+)-\chi(R_-)]. \tag{3.12}$$

Recall that the energy E entering the action (3.6) is an integral of motion whose value is equal to zero. Thus we can perform the standard procedure of excluding a degree of freedom in a canonical system. Equating the expression (3.7) to zero, we can express m_+ in terms of μ, v, and ξ. Substituting the result into Eq. (3.6), we get

$$-i\mathcal{I} = \int_{-\infty}^{+\infty} d\xi(\gamma^{-1}\mu\partial_\xi v - H), \tag{3.13}$$

$$H = -\mu v + \frac{\mu^2}{d}\phi(v) + \frac{|y|^2 L^\gamma}{D}[\chi(R_+)-\chi(R_-)]e^{\gamma\xi}, \tag{3.14}$$

$$\phi = (1+v)^{2-4/\gamma}[(1+v)^{2/\gamma-1}-1][(1+v)^{2/\gamma}-1]. \tag{3.15}$$

Here we kept main contributions over d only. In Eq. (3.14) we set $y^2=-|y|^2$ since as follows from Eqs. (2.21) and (3.4) y is an imaginary number. Extremum conditions for the action (3.13) read

$$\gamma^{-1}\frac{dv}{d\xi} = \frac{\partial H}{\partial\mu}, \quad \gamma^{-1}\frac{d\mu}{d\xi} = -\frac{\partial H}{\partial v}, \tag{3.16}$$

which are canonical equations for the variables μ and v in "time" ξ. Of course, Eqs. (3.16) could be obtained directly from the extremum conditions for the action (3.6).

To conclude, we reformulated the problem as follows: Find such a value of y that the solution of Eqs. (3.16) with the given y, being substituted into Eq. (3.12), reproduces the correct value of $\vartheta=-in/y$. Below we discuss the first and the most difficult part of the program that is solution of the system (3.16). Though it cannot be integrated exactly, we can solve the system approximately by asymptotic matching, which is enough to determine the structure functions S_n.

B. General structure of the instanton

The evolution of R_- in time ξ can be divided into three stages. During the first stage, starting at $\xi=-\infty$, both R_+ and R_- are much less than L and it is possible to substitute both $\chi(R_+)$ and $\chi(R_-)$ by $\chi(0)$. Then the last term in Eq. (3.14) is equal to zero. During the second stage $R_\pm\sim L$ and the last term in Eq. (3.14) is of importance. During the final stage, where $R_+\approx R_-\gg L$, one can again neglect the last term in Eq. (3.14). Note that only the second stage contributes to ϑ^2, which can be seen from Eq. (3.12). Since the Hamiltonian H [Eq. (3.14)] does not explicitly depend on time ξ during the first and third stages, its value is conserved there. Actually, the value of H is equal to zero during the third stage since $\mu\to 0$ and $R_+\approx R_-\gg L$ at $\xi\to+\infty$. On the other hand, during the first stage the value H_1 of the Hamiltonian function H is nonzero. Therefore, during the second stage the value of H diminishes and should finally reach zero when the trivial third stage starts. The value of H_1 as a function of n has to be established from the matching of the stages.

Now we are going to solve Eq. (3.16) for the first stage. Resolving the equation $H=H_1$ in terms of μ we get

$$\mu = \frac{d}{2\phi}(v-G), \tag{3.17}$$

$$G(v) = \pm\sqrt{v^2 + \frac{4H_1\phi}{d}}. \tag{3.18}$$

Then we find from Eq. (3.16)

$$\gamma^{-1}\frac{dv}{d\xi} = -G(v). \tag{3.19}$$

At $\xi\to-\infty$ (that is at small $|t|$) the function v should decrease with increasing ξ since $R_-\approx r$ and R_+ increases. To ensure the negative value of $dv/d\xi$ in Eq. (3.19) one should take the positive sign of the square root in Eq. (3.18), which leads to a positive value of G. The sign of G can be changed if during the evolution G turns into zero, which corresponds to the presence of a reversed point in the dependence of v on ξ.

Equation (3.19) enables one to find v as a function of ξ. Let us integrate the equation over ξ from $-\infty$ to some value. Then we get

$$\int_{\infty}^{v} dx\left[\frac{1}{x}-\frac{1}{G(x)}\right]=\ln\left[\frac{vR_{+}^{\gamma}}{r^{\gamma}}\right]. \tag{3.20}$$

To avoid difficulties related to infinite values of ξ and v at the initial point, we subtracted from G^{-1} its asymptote $G^{-1}(x)\approx 1/x$ at large x. This enforces the convergence of the integral (3.20) at large x. The constant of integration in Eq. (3.20) was established from the limit $v\to\infty$: Since the integral on the left-hand side of Eq. (3.20) tends to zero as v increases, the right-hand side of Eq. (3.20) should also tend to zero. This requirement is ensured by the r-dependent factor in Eq. (3.20) since $R_{+}^{\gamma}\approx r^{\gamma}/v$ at $v\to\infty$, as follows from the boundary condition $R_{-}(0)=r$ and Eq. (3.11). The left-hand side of Eq. (3.20) should be viewed as a contour integral, which determines its value in the case of the nonmonotonic behavior of v as a function of ξ.

Equation (3.20) allows us to establish a relation for the parameters characterizing the first stage. Let us consider the integral over the whole first stage. Then we should substitute $v=v_{*}$ in Eq. (3.20), where v_{*} is the value of v at the end of the first stage. The initial substage (where $v\gtrsim 1$) gives a constant of order unity in the integral on the left-hand side of Eq. (3.20) since $G(x)\approx x$ there. We neglect the contribution substituting $v\sim 1$ as the lower limit in the integral. Then the integral $\int dx/x$ produces just $\ln v$, which is canceled by the corresponding term on the right-hand side. Next, at the boundary between the first and the second stages $R_{-}\sim L$ since the pumping enters the game there. Therefore, with the logarithmic accuracy one can write

$$-\int_{1}^{v_{*}}\frac{dx}{G(x)}=\gamma\ln\left(\frac{L}{r}\right). \tag{3.21}$$

We see that there is a large parameter L/r in the argument of the logarithm on the right-hand side of Eq. (3.21). An analysis shows that due to this large parameter there are only two possibilities to satisfy the relation (3.21). Both of them are related to zeros of the function G because only near the points where G is small can the integral reach a large value. The first possibility is realized when G is zero only at $v=0$. In this case $v_{*}\ll 1$ and v is a monotonically decreasing function. The second possibility is that G is zero at some point $v=v_r$. That is just the reverse point where the derivative $dv/d\xi$ changes its sign; see Eq. (3.19). Then the integral on the left-hand side of Eq. (3.21) is determined by the vicinity of the point since G is small there.

A choice between the possibilities depend on the value of H_1. If $H_1>H_c$, where

$$H_c=-\frac{d\gamma^2}{8(2-\gamma)}, \tag{3.22}$$

then G cannot be zero (except for the point $v=0$). Then the integral on the left-hand side of Eq. (3.21) reaches its large value at $v\ll 1$. Substituting into Eq. (3.18) the asymptotic expression

$$\phi\approx\frac{2(2-\gamma)}{\gamma^2}v^2, \tag{3.23}$$

valid at $v\ll 1$, we can calculate the integral on the left-hand side of Eq. (3.21) with the logarithmic accuracy and find

$$\ln v_{*}=\gamma\sqrt{1-H_1/H_c}\ln\left(\frac{r}{L}\right). \tag{3.24}$$

We see that due to $r\ll L$, indeed $v_{*}\ll 1$.

In the opposite case $H_1<H_c$ the situation is more complicated. From the asymptotic expression

$$G^2(v)\approx\left[-\frac{8(2-\gamma)}{d\gamma^2}(H_c-H_1)+\frac{4-\gamma}{2\gamma}v\right]v^2, \tag{3.25}$$

valid at $v\ll 1$, we see that G is zero at $v=v_r$, where

$$v_r=\frac{16(2-\gamma)}{d\gamma(4-\gamma)}(H_c-H_1). \tag{3.26}$$

It is just the reverse point where the derivative $dv/d\xi$ changes its sign. Therefore, the sign of G is positive during the initial part of the first period and negative during the final one. Thus we should take the upper sign in Eq. (3.18) for the first part and the lower sign for the second part. The main contribution to the left-hand side of Eq. (3.21) is determined by the region near the reverse point $v-v_r\sim v_r$ where we can use the expression (3.25). The explicit integration gives

$$\sqrt{\frac{d}{2(2-\gamma)}}\frac{\pi}{\sqrt{H_c-H_1}}=\gamma\ln\frac{L}{r}. \tag{3.27}$$

Since the logarithm is large, H_1 is close to H_c and hence $v_r\ll 1$, as we implicitly assumed in the expression (3.25). Note that Eq. (3.27) does not fix the value of v_{*}, as it was for $H_1>H_c$.

Now we should extract additional relations that along with Eq. (3.24) or (3.27) will fix the instantonic solution and determine the final answer for the structure functions. It can be done by establishing the evolution during the second stage and by its subsequent matching with the first stage. Unfortunately, the procedure is rather lengthy and is individual for each particular case. We present the calculations in Appendix D.

C. Expressions for structure functions

Based on the reasoning given in the preceding subsection and on the calculations described in Appendix D, one can establish expressions for the structure functions from the relation (3.2). Here we enumerate basic results, referring the reader interested in technical details to Appendix D.

The case $H_1>H_c$ is realized if $n<n_c$ (see Appendix D 1), where

$$n_c=\frac{d\gamma}{2(2-\gamma)}. \qquad (3.28)$$

Calculating the action $\mathcal{I}$ and ϑ (see Appendix D 1) and substituting the result into Eq. (3.2), we obtain

$$S_n\sim\left(\frac{n}{\gamma}\frac{P_2C_1}{D}L^{\gamma}\right)^{n/2}\left(\frac{r}{L}\right)^{\zeta_n}. \qquad (3.29)$$

$$\zeta_n=\frac{n\gamma}{2}-\frac{(2-\gamma)n^2}{2d}. \qquad (3.30)$$

The quantity C_1 in the expression (3.29) is a constant of order unity, whose value depends on the shape of χ (that is, on the details of the pumping) and is consequently nonuniversal. Note that the r-independent factor in Eq. (3.29) is determined by the single-point root-mean-square value of the passive scalar

$$\theta_{\mathrm{rms}}^2\sim\frac{P_2}{D\gamma}L^{\gamma}. \qquad (3.31)$$

Comparing the expression (3.30) with Eq. (1.10), we see that they coincide under the conditions $n\gg1$ and $d\gg1$ that were implied in our derivation. Surprisingly, the n dependence of ζ_n given by Eq. (1.10) is correct not only in the limit (1.11) (that is, for $n\ll n_c$), but up to $n=n_c$, which is the boundary value for Eqs. (3.29) and (3.30).

A detailed consideration of the case $H_1>H_c$ is presented in Appendix . It shows that this possibility is realized at $n>n_c$. Then the scaling exponents ζ_n appear to be n independent and equal to the value

$$\zeta_c=\frac{d\gamma^2}{8(2-\gamma)}. \qquad (3.32)$$

The n-dependent numerical factors in S_n can be found in two limits: $n-n_c\ll n_c$ and $n\gg n_c$. The former case is discussed below, while in the latter case one can obtain (see Appendix D 2)

$$S_n\sim\left(\frac{n}{\gamma}\frac{P_2C_2}{D}L^{\gamma}\right)^{n/2}\left(\frac{r}{L}\right)^{\zeta_c}. \qquad (3.33)$$

The quantity C_2 in Eq. (3.33) is again a nonuniversal constant of order unity. The expression (3.33) corresponds to the factorized Gaussian PDF

$$\mathcal{P}(\vartheta)\sim\left(\frac{r}{L}\right)^{\zeta_c}\exp\left(-\frac{\gamma D\vartheta^2}{2C_2P_2L^{\gamma}}\right). \qquad (3.34)$$

Let us stress that when calculating $S_n\approx\langle|\vartheta|^n\rangle$ with the help of the PDF (3.34), the characteristic ϑ is of the order of the single-point root-mean-square value of the passive scalar (3.31) and the relatively small value of the result (3.33) compared to a single-point value is ensured only by the small r-dependent factor in Eq. (3.34). In Appendix D 3 we establish the inequality

$$\ln\frac{n}{d}<\gamma\ln\frac{L}{r}, \qquad (3.35)$$

which restricts the region where the expression (3.33) is correct. For larger n the character of the PDF essentially changes and it tends to a single-point PDF that is similar to Eq. (3.34) but does not contain the r-dependent factor.

Note that the cases $\gamma\ll1$ and $2-\gamma\ll1$ need a special analysis, which is performed in Appendix D 3. The answer (3.33) should be slightly corrected in the case $\gamma\ll1$ and keeps its form at $2-\gamma\ll1$.

We can treat the structure function S_n as a continuous function of n. Then the vicinity of the critical value $n=n_c$ requires a separate consideration, which is presented in Appendix . The main peculiarity that appears in the expressions for the structure functions is a critical dependence on n. The expression for the structure functions can be written as

$$S_n\sim\left[\frac{(n-n_c)^2}{\gamma n_c}\frac{P_2C_{\pm}}{D}L^{\gamma}\right]^{n_c/2}\left(\frac{r}{L}\right)^{\zeta_n}, \qquad (3.36)$$

which implies the condition $|n-n_c|\ll n_c$. The factors $C_{\pm}$ are nonuniversal constants of order unity which are different for the cases $n<n_c$ and $n>n_c$. The exponents ζ_n in the expression (3.36) are determined by Eq. (3.30) if $n<n_c$ and $\zeta_n=\zeta_c$ [Eq. (3.32)] if $n>n_c$. In the consideration made above we suggested that r/L is the smallest parameter of our theory. However, if $n\to n_c$, then $|n_c-n|$ starts to compete with r/L and at small enough n_c-n the consideration presented in Appendix is inapplicable. The criterion that determines the validity of Eq. (3.36) is established in Appendix D 4,

$$\gamma\ln\frac{L}{r}\gg\frac{n_c}{|n-n_c|}. \qquad (3.37)$$

We see that the first factor in Eq. (3.36) possesses the critical behavior proportional $|n-n_c|^{n_c}$ that is saturated in the narrow vicinity near $n=n_c$, where the condition (3.37) is violated. To avoid a misunderstanding, let us stress that despite the critical behavior, S_n remains a monotonically increasing function of n at a fixed L/r. This is obvious for $n>n_c$, whereas for $n<n_c$ it accounts for the stronger dependence on n of the second (r-dependent) factor in Eq. (3.36), which is guaranteed by the inequality (3.37).

We presented the results of the analysis based on the saddle-point approximation. The account of fluctuations on the background of our instanton could, in principle, change the results. Particularly the value of ζ_n could increase. Therefore, one should estimate the role of the fluctuations. The corresponding analysis is presented in Appendix B 2. It shows that for the condition (2.16) fluctuation effects are weak and cannot essentially change the results obtained.

CONCLUSION

We have performed an investigation of the structure functions in the Kraichnan model in the framework of the instantonic formalism. Though our approach is correct only for large dimensionalities of space, we observe a nontrivial picture, some peculiarities of which could be realized in a wider context. Below we discuss the results obtained.

We have established the n dependence of the scaling exponents, which are determined by the expression (3.30) for $n<n_c$ and remain the constant (3.32) for $n>n_c$, where n_c is defined by Eq. (3.28). Our results contradict the schemes proposed in [26,27]. The value (3.32) is different from and smaller than the constant obtained in [28], which can be considered really as an estimate from above. For $n\ll n_c$ our expression coincides with the answer obtained perturbatively [17,18] at large d. Surprisingly, the quadratic dependence of ζ_n on n is kept up to $n=n_c$. Such an n dependence of ζ_n is well known from the so-called log-normal distribution proposed by Kolmogorov [39].

The expressions (3.29) and (3.33) reveal the combinatoric prefactors in S_n that are characteristic rather of a Gaussian distribution. A natural explanation can be found in terms of zero mode ideology [15–19,41]. We know that for $n>2$ the main contribution to the structure function S_n in the convective interval is related to zero modes of the equation for the nth-order correlation function of the passive scalar. The exponents of the modes are determined by the equation (and could be very sensitive to the value of n), whereas numerical coefficients before the modes (determining their contribution to S_n) have to be extracted from matching on the pumping scale where the statistics of the passive scalar is nearly Gaussian. This explains the combinatoric prefactors in Eqs. (3.29) and (3.33). Probably the most striking feature of our results is the unusual behavior of S_n (treated as continuous functions of n) near $n=n_c$, which is determined by the expression (3.36).

Now we briefly discuss the interpretation of our results. The log-normal answer (3.29) and (3.30) can be obtained if we accept that for large fluctuations, giving the main contribution to the structure function S_n, the pumping is inessential and the fluctuation is smooth on the scale r. Then one obtains from Eq. (1.1) the equation for the passive scalar difference taken at the separation $\boldsymbol{r}$,

$$\partial_t \ln(\Delta\theta) = -\boldsymbol{v}\cdot\boldsymbol{r}/r^2,$$

where we substituted $\nabla\theta$ by $\Delta\theta/r$. We immediately get from this equation a log-normal statistics for $\Delta\theta$ that is a consequence of the central limiting theorem. The saturation at $n>n_c$ can be explained by the presence of quasidiscontinuous structures in the field θ making the main contribution to the high-order correlation functions of θ. Note also a similar nonanalytical behavior of ζ_n for Burgers' turbulence [7], which is explained by the presence of shocks in the velocity field. Although formally our scheme is applicable only in the limit $d\gamma\gg 1$, one can hope that the main features of our results persist for arbitrary values of the parameters. This hope is supported by [42], where a saturation of ζ_n was observed in numerical simulations of the Kraichnan model at $d=3$.

ACKNOWLEDGMENTS

We are grateful to G. Falkovich for valuable remarks and to M. Chertkov, R. Kraichnan, D. Khmelnitskii, and M. Vergassola for useful discussions. E.B. acknowledges support by the Joseph Meyerhoff Foundation and a grant from the Israel Science Foundation. V.L. acknowledges support from the Einstein Center, from the Minerva Center for Nonlinear Physics at the Weizmann Institute, and from the ENS-Landau Institute Twinning Programme.

APPENDIX A: SINGLE LAGRANGIAN SEPARATION

In this appendix we treat the statistics of a single Lagrangian separation defined by Eq. (2.4). The consideration will allow us to establish the relation (2.9) and also to clarify the condition (2.16).

1. Richardson law

A single Lagrangian difference $\boldsymbol{R}$ between two Lagrangian trajectories ϱ and $\varrho+\boldsymbol{R}$ is governed by the equation

$$\partial_t R_\alpha = w_\alpha \equiv v_\alpha(\varrho+\boldsymbol{R}) - v_\alpha(\varrho), \tag{A1}$$

with the correlation function

$$\begin{aligned}\langle w_\alpha(t_1) w_\beta(t_2)\rangle &= 2\mathcal{K}_{\alpha\beta}(\boldsymbol{R})\,\delta(t_1-t_2),\\ \mathcal{K}_{\alpha\beta}(\boldsymbol{R}) &= \frac{D}{d}R^{-\gamma}\left\{\frac{2-\gamma}{(d-1)}(R^2\delta_{\alpha\beta}-R_\alpha R_\beta)+R^2\delta_{\alpha\beta}\right\},\end{aligned} \tag{A2}$$

following from Eq. (1.6). First of all, we get from Eqs. (2.10) and (2.11)

$$\langle \zeta(t_1)\zeta(t_2)\rangle = \frac{2D}{d}R^\gamma \delta(t_1-t_2), \tag{A3}$$

where in accordance with Eq. (2.8) $\zeta=\gamma^{-1}\partial_t R^\gamma$. Then

$$R(t-\Delta t) - R(t) \approx R^{1-\gamma}(t)\int_t^{t-\Delta t} d\tau\,\zeta(\tau), \tag{A4}$$

where we believe $\Delta t>0$ to be a small time interval (recall that we treat evolution backward in time). Averaging over the velocity statistics on the interval from t to $t-\Delta t$, we get from Eqs. (A3) and (A4)

$$\langle [R(t-\Delta t)-R]^2\rangle = \frac{2D}{d}R^{2-\gamma}, \tag{A5}$$

where $R\equiv R(t)$.

Let us write now

$$R_\alpha(t-\Delta t) = R_\alpha(t) + \int_t^{t-\Delta t} d\tau\, w_\alpha[\tau, R_\alpha(\tau)], \tag{A6}$$

which is the direct consequence of Eq. (A1). Then we find from Eq. (A6) in the approximation needed for us

$$\begin{aligned}&R^2(t-\Delta t) - R^2(t)\\ &\quad\approx 2R_\alpha \int_t^{t-\Delta t} d\tau\, w_\alpha(\tau,\boldsymbol{R})\\ &\qquad + \int\int_t^{t-\Delta t} d\tau\, d\tau'\, w_\alpha(\tau,\boldsymbol{R}) w_\alpha(\tau',\boldsymbol{R})\\ &\qquad + R_\alpha \frac{\partial}{\partial R_\beta}\int_t^{t-\Delta t} d\tau\, w_\alpha(\tau,\boldsymbol{R})\int_t^\tau d\tau'\, w_\beta(\tau',\boldsymbol{R}),\end{aligned}$$

where again $\boldsymbol{R}\equiv\boldsymbol{R}(t)$ and we used the incompressibility condition $\partial \boldsymbol{w}/\partial\boldsymbol{R}=0$. Averaging the expression over the velocity statistics on the interval from t to $t-\Delta t$, we get

$$\langle R^2(t-\Delta t)-R^2\rangle\approx 2\frac{D}{d}(d+2-\gamma)R^{2-\gamma}\Delta t, \qquad \text{(A7)}$$

where we used the expressions (A2) and taken into account $\partial\mathcal{K}_{\alpha\beta}(\boldsymbol{R})/\partial R_\alpha=0$. Then we obtain from Eqs. (A5) and (A7)

$$\langle R(t-\Delta t)-R\rangle\approx\frac{D}{d}(d+1-\gamma)R^{1-\gamma}\Delta t. \qquad \text{(A8)}$$

Next, we get from the definition (2.8)

$$\langle R^\gamma(t-\Delta t)-R^\gamma\rangle=-\gamma\langle\zeta\rangle\Delta t. \qquad \text{(A9)}$$

Expanding here the difference up to second order over $R(t-\Delta t)-R$ and substituting then Eqs. (A5) and (A8) we find finally

$$\langle\zeta\rangle=-D. \qquad \text{(A10)}$$

Note that the average is negative, which is a consequence of considering an evolution backward in time.

The average value $\langle\zeta\rangle$ is obviously the same for all the Lagrangian separations. Therefore, we arrive at Eq. (2.9), leading then to Eq. (2.22), which is a manifestation of the Richardson law.

2. Simultaneous PDF

The Kraichnan model admits a closed description of the simultaneous probability distribution function $\mathcal{P}(\boldsymbol{R})$ for any single Lagrangian difference $\boldsymbol{R}$. The point is that Eq. (A1) in this case can be considered as a stochastic process with white noise on the right-hand side. It is well known how to obtain the equation for $\mathcal{P}(\boldsymbol{R})$ in the situation. Using the expression (A2) we get

$$\partial\mathcal{P}/\partial|t|=\mathcal{K}_{\alpha\beta}(\boldsymbol{R})\boldsymbol{\nabla}_\alpha\boldsymbol{\nabla}_\beta\mathcal{P}. \qquad \text{(A11)}$$

The same problem for $\gamma=2/3$ was considered by Kraichnan [43], who obtained the asymptotic behavior of $\mathcal{P}$ at large times t. Here we present a straightforward generalization of Kraichnan's scheme for arbitrary γ.

Suppose that at $t=0$ the PDF is $\mathcal{P}=R_0^{1-d}\delta(R-R_0)$; the initial condition corresponds to a fixed value of $|\boldsymbol{R}|$ at $t=0$. The above expression implies the normalization condition $\int dR\, R^{d-1}\mathcal{P}(t,R)=1$. Solutions corresponding to other initial conditions can be expressed via this fundamental solution since Eq. (A12) is linear. Due to isotropy, $\mathcal{P}$ will be a function of R only. Then Eq. (A11) is rewritten as

$$\frac{d}{D}\frac{\partial\mathcal{P}}{\partial|t|}=\frac{1}{R^{d-1}}\frac{\partial}{\partial R}\left(R^{d+1-\gamma}\frac{\partial}{\partial R}\mathcal{P}\right). \qquad \text{(A12)}$$

Of course Eq. (A12) can also be obtained directly from Eqs. (A3) and (A10).

Performing a Laplace transform of Eq. (A12), one obtains

$$\mathcal{P}(t,R)=\int_{A-i\infty}^{A+i\infty}\frac{d\lambda}{2\pi i}\exp\left(\frac{D}{d}\lambda|t|\right)\mathcal{S}(\lambda,R), \qquad \text{(A13)}$$

where all the singularities of $\mathcal{S}(\lambda)$ have to be to the left of the integration contour. The function $\mathcal{S}$ in Eq. (A13) satisfies the equation

$$\lambda\mathcal{S}-\frac{1}{R^{d-1}}\frac{\partial}{\partial R}\left(R^{d+1-\gamma}\frac{\partial}{\partial R}\mathcal{S}\right)=\frac{1}{R_0^{d-1}}\delta(R-R_0).$$

The equation can be solved separately in the regions $R<R_0$ and $R>R_0$ where we deal with the homogeneous equation and then the matching conditions at $R=R_0$ give us coefficients. Assuming suitable boundary conditions ($\mathcal{S}$ is finite at $R\to0$ and at $R\to\infty$) we get

$$\mathcal{S}(\lambda,R)=\frac{2}{\gamma}(RR_0)^{-d/2+\gamma/2}\begin{cases} I_\nu\left(\frac{2\sqrt{\lambda}}{\gamma}R^{\gamma/2}\right)K_\nu\left(\frac{2\sqrt{\lambda}}{\gamma}R_0^{\gamma/2}\right) & \text{if } R<R_0\\ K_\nu\left(\frac{2\sqrt{\lambda}}{\gamma}R^{\gamma/2}\right)I_\nu\left(\frac{2\sqrt{\lambda}}{\gamma}R_0^{\gamma/2}\right) & \text{if } R>R_0. \end{cases} \qquad \text{(A14)}$$

Here $\nu=d/\gamma-1$, I_ν is the modified Bessel function, and K_ν is the McDonald function.

If we are interested in the asymptotic behavior of $\mathcal{P}(t,R)$ at large times $|t|\gg dR_0^\gamma/D\gamma^2$, then R near the maximum of $\mathcal{P}$ satisfies $R\gg R_0$. In addition, in the integral (A13) the characteristic value $\lambda_{\mathrm{char}}\propto|t|^{-1}$ is small. Thus we can take only the second term in Eq. (A14) (corresponding to $R>R_0$) and substitute the first term of the expansion $I_\nu(z)\simeq\Gamma^{-1}(1+\nu)(z/2)^\nu$. Then the integral (A13) can be taken explicitly and we obtain

$$\mathcal{P}(t,R)R^{d-1}dR=\frac{1}{\Gamma(1+\nu)}d\xi\,\xi^\nu\exp(-\xi), \qquad \text{(A15)}$$

$$\xi=\frac{dR^\gamma}{D\gamma^2|t|}, \qquad \text{(A16)}$$

which gives the asymptotic self-similar behavior of PDF. We see that $\ln\mathcal{P}\propto-R^\gamma$. It is interesting that the asymptotic PDF is Gaussian at $\gamma=2$ for any d. As we move γ from two to zero, the PDF is getting more and more non-Gaussian, reaching an extreme non-Gaussian nature (log-normality) at $\gamma=0$. Note an obvious consequence of Eqs. (A15) and (A16):

$$\langle R^{\mu\gamma}\rangle\propto|t|^\mu \qquad \text{(A17)}$$

for arbitrary μ. The asymptotic law (A17) is a manifestation of the Richardson law.

It is not very difficult to calculate first moments using the expression (A15):

$$\langle R^2\rangle=\left(\frac{D\gamma^2|t|}{d}\right)^{2/\gamma}\frac{\Gamma(1+\nu+2/\gamma)}{\Gamma(1+\nu)},\qquad\text{(A18)}$$

$$\langle R^4\rangle=\left(\frac{D\gamma^2|t|}{d}\right)^{4/\gamma}\frac{\Gamma(1+\nu+4/\gamma)}{\Gamma(1+\nu)}.\qquad\text{(A19)}$$

In the large d limit the expressions (A18) and (A19) give

$$\langle R^2\rangle=(D\gamma|t|)^{2/\gamma},\qquad\text{(A20)}$$

$$\frac{\langle R^4\rangle-\langle R^2\rangle^2}{\langle R^2\rangle^2}=\exp\left(\frac{4}{\gamma d}\right)-1,\qquad\text{(A21)}$$

irrespective of the value of γ. We see that fluctuations of R^2 are small if $\gamma d\gg 1$ and in the opposite limit $\gamma d\ll 1$ the cumulant of R^2 is much larger than $\langle R^2\rangle^2$. We conclude that the inequality (2.16) is just the condition at which R^γ only weakly fluctuates near the value (2.22).

APPENDIX B

Here we discuss the applicability conditions of our scheme. First, the triangular inequality (2.15) should be satisfied for our instantonic solution. Second, fluctuations on the background of our instanton should be weak.

1. Triangle inequality

Here we discuss the triangular inequality (2.15) that was ignored in our saddle-point calculations. We argue that if the parameter $d\gamma$ is large, the inequality (2.15) is satisfied for the instanton solution. In other words, we can say that calculating the path integral (2.13), we can dismiss the restriction supplied by Eq. (2.15) because the contribution from the regions where the inequality is violated is small in this limit.

Let us recall that the inequalities (2.15) are obviously satisfied for both the initial condition (2.14) and the asymptotic behavior (2.22). Therefore, they could be violated only for times of the order of the instanton lifetime. Our project will be as follows. First, we check that the inequality (2.15) holds for the instanton solution if we consider the instantonic equations in the main order over $1/d$. Then we will show that next-order terms over $1/d$ considered in Appendix C can lead to violation of the inequality if $d\gamma\lesssim 1$. Generally, this procedure is complicated and here we present only some calculations that serve as a basis for our conclusions.

As a first step we have to restore the field R_{12} in the whole space, that is, for any two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. This can be done as follows. Let us consider Eq. (2.18). Substituting m_{12} in the form (3.3) we get

$$\gamma^{-1}\partial_t R_{12}^{\gamma}+D=-\frac{Dm_-}{2d}\,\frac{(R_{1-}^{2-\gamma}+R_{2+}^{2-\gamma}-R_{1+}^{2-\gamma}-R_{2-}^{2-\gamma})(R_{1-}^2+R_{2+}^2-R_{1+}^2-R_{2-}^2)}{R_{12}^{2-\gamma}R_-^{2-\gamma}}.\qquad\text{(B1)}$$

We kept only the main term over $1/d$ in Q [Eq. (2.11)]. In Eq. (B1) we introduced auxiliary fields

$$R_{1+}=R(\boldsymbol{r}_1,\boldsymbol{r}/2),\quad R_{1-}=R(\boldsymbol{r}_1,-\boldsymbol{r}/2).\qquad\text{(B2)}$$

They satisfy the closed system of equations

$$\gamma^{-1}\partial_t R_{1+}^{\gamma}+D=-\frac{Dm_-}{2d}\,\frac{(R_{1-}^{2-\gamma}+R_{+}^{2-\gamma}-R_{1+}^{2-\gamma}-R_{-}^{2-\gamma})(R_{1-}^2+R_{+}^2-R_{1+}^2-R_{-}^2)}{R_{1+}^{2-\gamma}R_-^{-\gamma}},\qquad\text{(B3)}$$

$$\gamma^{-1}\partial_t R_{1-}^{\gamma}+D=-\frac{Dm_-}{2d}\,\frac{(R_{1-}^{2-\gamma}+R_{-}^{2-\gamma}-R_{1+}^{2-\gamma}-R_{+}^{2-\gamma})(R_{1-}^2+R_{-}^2-R_{1+}^2-R_{+}^2)}{R_{1-}^{2-\gamma}R_-^{2-\gamma}},\qquad\text{(B4)}$$

which can be obtained from Eq. (B1). Thus one can restore the function R_{12} in two steps: first solving Eqs. (B3) and (B4) and then substituting the functions $R_{1\pm}$ into Eq. (B1). Though at each step one should solve ordinary differential equations, this is a hard program and we are not able to perform it entirely. Nevertheless, we can examine some crucial cases.

First of all, using the above scheme one can establish the behavior of the field R_{12} where $\boldsymbol{r}_{1,2}$ are close to $\pm\boldsymbol{r}/2$. The analysis shows that the field R_{12} is smooth near the points. This is the justification of our reduction procedure leading to the effective action (3.6).

Simple consideration shows that the most dangerous geometry that could lead to violation of the triangular inequalities (2.15) is realized if two of the three points are close to $\pm\boldsymbol{r}/2$, whereas the third one (say, $\boldsymbol{r}_1$) lies in the middle between the points. In this case $R_{1+}=R_{1-}$ and the triangular inequality tells in this case that the difference $2R_{1-}-R_{-}$ should be positive. To prove the inequality let us write the equation for this quantity

$$\frac{d(2R_{1-}-R_{-})}{dt}=-D(2R_{1-}^{1-\gamma}-R_{-}^{1-\gamma})-\frac{Dm_{-}}{d}\frac{(R_{+}^{2-\gamma}-R_{-}^{2-\gamma})(R_{+}^{2}-R_{-}^{2})}{R_{-}^{2-\gamma}}\frac{(R_{-}-2R_{1-})}{R_{-}R_{1-}}. \tag{B5}$$

It is easy to show that at small times the quantity $2R_{1-}-R_{-}$ increases. Suppose that at some moment of time it becomes zero. Then the second term on the right-hand side of Eq. (B5) is zero, while the first one is negative. This means that the difference $2R_{1-}-R_{-}$ increases (recall that we move backward in time), though it should approach zero from the positive side. The contradiction proves that $2R_{1-}-R_{-}$ is always positive.

Let us now restore the terms subleading over $1/d$. Adding the terms to Eq. (B1), we can repeat our consideration. Again, we should consider the same geometry as above. Writing the equation for $2R_{1-}-R_{-}$, we are convinced that in addition to the two terms presented in Eq. (B5) one should take into account also the term of the next order over $1/d$. This term is nonzero when $2R_{1-}=R_{-}$; therefore, it starts to compete with the leading term $-DR_{-}^{1-\gamma}(2^{\gamma}-1)$ if $2R_{1-}-R_{-}$ is small. In this case the leading term is proportional to $2^{\gamma}-1$, that is, to γ at small γ. Therefore, if γd is not large, we cannot make a definite conclusion about the sign of the difference. Thus we arrive at the inequality (2.16) formulated in the main text. The crucial cases investigated above make us confident that the triangular inequality holds for our instanton in the whole space provided the inequality (2.16) is satisfied.

2. Fluctuations

Here we extend the analysis of the fluctuations presented in the Sec. II B that is suitable for all the points excluding vicinities of $\pm\boldsymbol{r}/2$. The reason is that the quantity v introduced in Eq. (3.11) is small during some stages of the evolution. Therefore, $R_{+}=R(\boldsymbol{r}/2,\boldsymbol{r}/2)$ and $R_{-}=R(\boldsymbol{r}/2,-\boldsymbol{r}/2)$ are almost equal to each other. Thus we should check that fluctuations do not destroy this proximity. Having the problem in mind, we will assume $v\ll 1$ below.

We can take the effective action (2.23) as the starting point of our analysis. It will be enough for our purpose to examine fluctuation effects in the harmonic approximation. Therefore, we should expand the effective action (2.23) up to second order over the fluctuations δR_{12}^{γ} and δm_{12}. The first-order term of the expansion vanishes due to our saddle-point equations. The second-order term can be written as

$$\begin{aligned} i\mathcal{I}^{(2)}=&\, i\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2\, \delta m_{12}\gamma^{-1}\partial_t \delta R_{12}^{\gamma} \\ &-\frac{D}{d}\int dt\, d^4\boldsymbol{r}\, Q_{12,34}\delta m_{12}\delta m_{34} \\ &+\frac{1}{2}\int dt\, d\boldsymbol{r}_1 d\boldsymbol{r}_2 |y|^2 \chi^{(2)}(R_{12})\beta(\boldsymbol{r}_1)\beta(\boldsymbol{r}_2) \\ &-\frac{D}{d}\int dt\, d^4\boldsymbol{r}\{2\,\delta Q_{12,34}m_{12}\delta m_{34}+Q_{12,34}^{(2)}m_{12}m_{34}\}, \end{aligned} \tag{B6}$$

where $d^4\boldsymbol{r}=d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{r}_3 d\boldsymbol{r}_4$. Here R_{12} is the field corresponding to the instanton (recall that the way to restore it in the whole space was discussed in Appendix B 1), m_{12} is determined by the expression (3.3), and $\chi^{(2)},Q^{(2)}$ are the second-order terms in the expansion over δR_{12}^{γ}. As follows from Eqs. (3.1) and (3.3), the last two terms in Eq. (B6) are relevant only for the points close to $\pm\boldsymbol{r}/2$. Therefore, in the general case we can disregard these terms and return to the estimate (2.26), which can be obtained if only the two first terms in the action (B6) are kept. However, we are interested just in the behavior of R_{12} when the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are close to $\pm\boldsymbol{r}/2$. In this case a special analysis is needed.

First note that short-scale fluctuations of R_{12} are weak due to the restriction (2.15), which makes the amplitude of the fluctuations proportional to the scale. In other words, we should deal only with smooth functions R_{12}. Next, due to the presence of the second term in the effective action (B6) fluctuations of m_{12} are relatively suppressed for points that are not very close to $\pm\boldsymbol{r}/2$. That is a consequence of the $\boldsymbol{r}$ dependence of $Q_{12,34}$ [Eq. (2.11)], which in the case $v\ll 1$ has deep minima if the points $\boldsymbol{r}_i$ are close to $\pm\boldsymbol{r}/2$ (more precisely some linear combinations have deep minima; they just determine the structure of strong fluctuations of m). Therefore, relevant fluctuations of R_{12} can be estimated in terms of the expression (3.3) and, consequently, in terms of the reduced action (3.6). Since in the main approximation over d the terms with φ_2 and φ_3 can be neglected in Eq. (3.7), the integration over m_{+} can be done explicitly. That leads simply to fixing the expression (3.10) and reduces the action to the form (3.13).

So we should estimate fluctuations of v and μ starting from the action (3.13) with the Hamiltonian (3.14). Actually, we should check the validity of the semiclassical approximation for this system with one degree of freedom. It is more convenient to perform this conventional procedure in terms of the canonically conjugated variables p and q where $p=\mu v/\gamma$ and $v=\exp(q)$. Then the Hamiltonian (3.14) is rewritten as

$$H=-\gamma p+\frac{2(2-\gamma)}{d}\left[1-\frac{4-\gamma}{2\gamma}\exp(q)\right]p^2-\frac{|y|^2}{D\gamma}R_{+}^{1+\gamma}\chi'(R_{+})\exp(q), \tag{B7}$$

where we took into account $v\ll 1$. The subsequent analysis shows that the semiclassical approximation is broken only in the vicinity of the reverse point v_r that exists at $n>n_c$. Let us estimate this vicinity. Near the reverse point the last term in Eq. (B7) can be neglected and resolving the relation $H=H_1$ we get

$$p=\frac{n_c}{2}\left[1\pm\sqrt{\frac{4-\gamma}{2\gamma}}\sqrt{\exp(q)-v_r}\right],$$

where we substituted Eq. (3.26). The semiclassical approximation is broken if $p^{-2}dp/dq\sim 1$, which gives

$$v-v_r\sim\frac{v_r^2}{n_c^2}\ll v_r\,.$$

Since the main contribution to the integrals such as Eq. (3.21) is made at $v-v_r\sim v_r$, the above narrow vicinity (where the semiclassical approximation is broken) is irrelevant for our results.

Of course, the above analysis of the fluctuations on the background of our instanton is not exhaustive. Nevertheless, we believe that the arguments presented demonstrate the weakness of the fluctuations.

APPENDIX C

As was mentioned in Sec. III A, keeping only the main term over $1/d$ in the action (3.6) is potentially dangerous because of a possible conflict of the limits $d\gg 1$ and $L/r\gg 1$. If this were the case, our results would not be applicable deep in the convective interval, where the ratio r/L is very small. Fortunately, this is not the case. In this appendix we will consider the equations, following from the action (3.6), not neglecting terms subleading over $1/d$. We show that r/L does not interfere with $1/d$ and the scheme presented in Sec. III is reproduced with minor modifications. Repeating the scheme, we obtain expressions for the anomalous exponents ζ_n in this formulation. To avoid a misunderstanding let us stress that our scheme is correct only in the limit $d\gamma > 1$ (see Appendix B 1). Therefore, one could consider this appendix only as a demonstration of the absence of the aforementioned conflict of limits.

Below we will assume that $D=1$, $L=1$, and $P_2=1$, which can be done by rescaling time t, the coordinates $\boldsymbol{r}$, and the passive scalar θ. To restore the full answers one should add simply the factor P_2/DL^γ to ϑ^2 in all expressions.

Extremum conditions for the action (3.6) give four equations of motion for the quantities $R_\pm$ and $m_\pm$. As before, the boundary conditions to the equations are $R_+=0$ and $R_-=r$ at $t=0$ and $m_\pm\to 0$ at $t\to-\infty$. The equations are canonical, with the Hamiltonian given by Eq. (3.7). Since it does not depend on time explicitly, the energy is conserved. Moreover, $E=0$, which is a consequence of the asymptotic behavior $m_\pm\to 0$ of the instantonic solution at $t\to-\infty$.

As it is known from classical mechanics, for a Hamiltonian system that has an integral of motion one can reduce the number of degrees of freedom by one. In our case, we can pass from the system of four canonical equations to that of two. Let us express, say, m_+ via the other variables with the help of the conservation law $E=0$:

$$m_+\varphi_3=\alpha-m_-\varphi_2-\sqrt{(m_-\varphi_2-\alpha)^2-\varphi_3[m_-^2\varphi_1+2\alpha(m_-+|y|^2U)]},\tag{C1}$$

where

$$U=\chi(R_+)-\chi(R_-),\qquad \alpha=\frac{2d(d-1)}{2-\gamma}.\tag{C2}$$

The sign in front of the square root in Eq. (C1) should be minus to ensure the correct behavior of R_+ at small time.

Let us make the substitution (3.11). Then we obtain a generalization of Eqs. (3.13) and (3.14),

$$-i\mathcal{I}=\int d\xi[\gamma^{-1}\mu\partial_\xi v-H],\tag{C3}$$

$$H=-\mu(1+v)+\frac{\alpha-\mu\phi_2}{\phi_3}-\phi_3^{-1}\sqrt{(\mu\phi_2-\alpha)^2-\phi_3\{\mu^2\phi_1+2\alpha[\mu+|y|^2Ue^{\gamma\xi}]\}}.\tag{C4}$$

Note that

$$U=\chi(e^\xi)-\chi[(1+v)^{1/\gamma}e^\xi],\qquad m_+=\frac{H+\mu(1+v)}{e^{\gamma\xi}}.$$

We introduced here the notation

$$\phi_1=-(1+v)^{2-4/\gamma}\left[1+\frac{[2(1+v)^{2/\gamma}-1]^2}{1+v}\right]+\frac{4(d+1-\gamma)}{2-\gamma}[(1+v)^{1-2/\gamma}-1][(1+v)^{1-2/\gamma}-1-v],$$
$$\phi_2=2+\frac{1}{(1+v)^{2/\gamma-1}}-\frac{1}{(1+v)^{2/\gamma}},\quad \phi_3=-\frac{2+v}{1+v}. \tag{C5}$$

One can derive also a generalization of Eq. (3.12),

$$\vartheta^2=\int_{-\infty}^{\infty}\frac{2n\alpha\, e^{\gamma\xi}U\,d\xi}{\sqrt{(\mu\phi_2-\alpha)^2-\phi_3\{\mu^2\phi_1+2\alpha[\mu+|y|^2Ue^{\gamma\xi}]\}}}.$$

As before (see Sec. III A) we can divide the evolution into three stages. Let us analyze the first stage. Since $U=0$ there, the quantity H does not explicitly depend on ξ and therefore its value (which we designate H_1) is conserved during the first stage. Then from the relation (C4) we can express μ via v as

$$\mu=\frac{\alpha v-H_1S_4\pm F_1}{S_1}, \tag{C6}$$

$$F_1=\sqrt{H_1^2S_3+2\alpha H_1S_2+\alpha^2v^2}. \tag{C7}$$

We introduced the shorthand notation

$$S_1=\phi_1+2(1+v)\phi_2+\phi_3(1+v)^2, \tag{C8}$$

$$S_2=\phi_1+\phi_2+(1+v)(\phi_2+\phi_3), \tag{C9}$$

$$S_3=\phi_2^2-\phi_1\phi_3, \tag{C10}$$

$$S_4=\phi_2+(1+v)\phi_3. \tag{C11}$$

Then we can derive an equation for v,

$$\gamma^{-1}\partial_\xi v=\frac{\pm S_1F_1}{\alpha S_2+H_1S_3\mp S_4F_1}\equiv -G(v). \tag{C12}$$

This equation is the direct generalization of Eq. (3.19). To ensure the finite value of the action (C3), one should take the lower signs in Eqs. (C6) and (C12). A solution of Eq. (C12) with the correct boundary condition is given by Eq. (3.20), where G should be substituted from Eq. (C12).

Let us establish the behavior of G at small v, which is important for the description of the first stage. To do that, we should take into account the relations

$$\phi_1=-2-3v+\frac{8d-4\gamma-\gamma^2}{\gamma^2}v^2,$$

$$\phi_2=2+v-\frac{2}{\gamma}v^2,\quad \phi_3=-2+v-v^2,$$

$$S_1=\frac{8(d-\gamma)}{\gamma^2}v^2,\quad S_2=\frac{8(d-\gamma)}{\gamma^2}v^2,$$

$$S_3=\frac{16(d-\gamma)}{\gamma^2}v^2,\quad S_4=-\frac{2}{\gamma}v^2,$$

valid at $v\ll 1$. Note that all the functions S_i have the homogeneous behavior proportional to v^2 at small v. Therefore, starting from the relation (3.20), we can find the same estimates (3.24) and (3.26) for v_* and v_r with small corrections of the order $1/d$.

Next we should analyze the second stage and match it with the first stage, that is, a repetition of the procedure described in Appendix D. As a result we find the exponents

$$\zeta_n=\frac{1}{2}[\alpha-\sqrt{\alpha^2-2\alpha\gamma n+4(d-\gamma)n^2}], \tag{C13}$$

$$\zeta_c=\frac{\alpha}{2}\left(1-\sqrt{1-\frac{\gamma^2}{4(d-\gamma)}}\right). \tag{C14}$$

In the limit $\gamma d\gg 1$ we recover the previous results (3.30) and (3.32). To avoid a misunderstanding, let us stress that the expressions (C13) and (C14) cannot be used to establish $1/d$ corrections to the exponents (3.30) and (3.32) since contributions of the same order, related to the triangular inequalities (2.15) and fluctuations, are unknown.

APPENDIX D

In this appendix we present a consideration of the second stage and matching conditions for the instanton solution. The procedure appears to depend strongly on the order n of the structure function. Therefore, we consider different cases separately. The designations used below were introduced in Sec. III.

1. Intermediate tail

Let us first consider the case where the behavior of the function $v(\xi)$ is monotonic during the whole first stage. As was demonstrated in the main text, this case is realized if $H_1>H_c$ and the variable v is small at the end of the first stage. We will show that the variable remains small also during the second stage, going to zero at the third stage. Therefore, the evolution of v during the last substage of the first stage and during the second and the third stages can be described in terms of the Hamiltonian (3.14), where the condition $v \ll 1$ is utilized, which simplifies the analysis essentially.

In the limit $v \ll 1$ we get from Eqs. (3.14) and (3.15)

$$H = -\mu v + \frac{2(2-\gamma)}{d\gamma^2}\mu^2 v^2 - \frac{|y|^2}{\gamma D} R_+^{1+\gamma}\chi'(R_+)v. \tag{D1}$$

We see that the first two terms in the expression (D1) depend only on μv. This is the reason why the equation for the quantity

$$\frac{d(\mu v)}{d\xi} = \frac{|y|^2}{D} R_+^{1+\gamma}\chi'(R_+)v, \tag{D2}$$

that can be derived from Eqs. (D1) and (3.16) contains on the right-hand side only the term proportional to $\chi'(R_+)$. The term can be neglected if either $R_+ \ll 1$ or $R_+ \gg 1$, that is, during the first and during the third stages. Therefore, the quantity μv is conserved there. Recall that due to the boundary condition, $\mu = 0$ at $\xi \to \infty$ and consequently $\mu v = 0$ during the third stage.

Integrating the relation (D2) from any ξ corresponding to the last substage of the first stage up to $+\infty$, we get the integral relation

$$\mu v = -\frac{|y|^2}{D}\int d\xi\, R_+^{1+\gamma}\chi'(R_+)v, \tag{D3}$$

which is correct for the first stage if $v \ll 1$. It is instructive to compare Eq. (D3) with the relation

$$\vartheta^2 = -\frac{2n}{\gamma D}\int d\xi\, R_+^{1+\gamma}\chi'(R_+)v, \tag{D4}$$

which can be obtained from Eq. (3.12) at $v \ll 1$. Recalling also the relation $y = -in/\vartheta$ [Eq. (2.21)], we get

$$\mu v = \frac{\gamma n}{2}. \tag{D5}$$

Substituting Eq. (D5) into Eq. (D1) and neglecting the term with χ', we find that the value of the Hamiltonian during the first stage is

$$H_1 = -\frac{\gamma n}{2} + \frac{2-\gamma}{2d}n^2. \tag{D6}$$

During the second stage μv diminishes from $\gamma n/2$ to zero. Therefore, the equation

$$\frac{d\ln v}{d\xi} = \gamma\left(-1 + \frac{4(2-\gamma)}{d\gamma^2}\mu v\right)$$

shows that the quantity v does not vary essentially during the second stage. Then we obtain from Eqs. (D4) and (2.21) the estimates

$$\vartheta^2 \sim v_* L^\gamma \frac{nP_2}{\gamma D}, \quad |y|^2 \sim \frac{n\gamma D}{L^\gamma P_2 v_*}. \tag{D7}$$

Recall that v_* is the value of v at the end of the first stage.

Substituting Eqs. (3.23) and (D6) into Eq. (3.18) we get

$$G = v(1 - n/n_c), \tag{D8}$$

where n_c is defined by Eq. (3.28). Recall that the monotonic behavior of v implies $G > 0$ and therefore the expression in Eq. (D8) is correct if $n < n_c$. This means that for $n > n_c$ the instantonic solution of the considered type does not exist and we should look for another possibility. We postpone the problem to the next subsection and continue to analyze the monotonic v regarding $n < n_c$.

Substituting Eq. (D6) into Eq. (3.24) we get

$$\ln\frac{1}{v_*} = (1 - n/n_c)\gamma \ln\frac{L}{r}. \tag{D9}$$

Next we should find the leading contribution to the action. This contribution is made mainly by the first stage producing a large logarithm. Therefore, we can write, using Eq. (3.13),

$$i\mathcal{I} \approx \gamma^{-1}\int_{v_*}^1 dv\left(\mu + \frac{H_1}{G}\right).$$

Substituting here Eqs. (D5), (D6), and (D8), we find with the same logarithmic accuracy

$$i\mathcal{I} = -\frac{(2-\gamma)n^2}{2d}\ln\frac{L}{r}. \tag{D10}$$

Finally, we can determine the structure functions S_n [Eq. (1.2] in accordance with the formula (3.2). Collecting Eqs. (D7), (D9), and (D10) we obtain the expressions (3.29) and (3.30).

2. Remote tail

Let us proceed to discuss the character of the instantonic solution at $n > n_c$. We can solve the corresponding equations in two limiting cases: $n \gg n_c$ and $n - n_c \ll n_c$. The latter case is considered in Appendix D 4. In this subsection we accept $n \gg n_c$; the inequality ensures also $v_* \gg 1$ (recall that v_* is the value of v at the end of the first stage). Due to the condition $v_* \gg 1$, the last substage of the first stage and the second stage can be examined in terms of the Hamiltonian

$$H = -\mu v + \frac{\mu^2}{2d}v + \frac{|y|^2}{D}R_+^\gamma[\chi(R_+) - \chi(R_+ v^{1/\gamma})], \tag{D11}$$

which follows from Eqs. (3.11), (3.14), and (3.15) at $v \gg 1$. At the end of the first stage $R_- \sim L$, whereas $R_+ \approx R_-/v_*^{1/\gamma} \ll L$. The second of Eqs. (3.16) shows that μ, which is equal to zero at $R_- > L$, varies essentially at $R_- \sim L$. To find the value of μ at $R_- < L$, we will use the relation

$$\frac{d}{d\xi}[(H + \mu v)R_+^{-\gamma}] = \frac{|y|^2}{D}\chi'(R_+)R_+, \tag{D12}$$

which follows from Eqs. (3.16) and (D11). Actually Eq. (D12) is the equation for m_+ that can be obtained from Eqs. (3.6), (3.7), and (3.9) under the same conditions that led to Eq. (D11). Since $R_+ \ll R_- \sim L$ during the last substage of the first stage and the second stage, the right-hand side of Eq. (D12) can be neglected and we get a conservation law for $(H+\mu v)R_+^{-\gamma}$. Equating the values of that quantity at $R_- < L$ and at $R_- > L$, we find that at the end of the first stage

$$\mu_* v_* \sim \sqrt{\frac{d|y|^2 P_2}{D} L^\gamma}. \tag{D13}$$

Now let us return to the relation (3.4). Since $R_- \sim L \gg R_+$ at the end of the first stage the main contribution to the integral (3.4) is made when already $R_- > L$ and R_+ increases to L. Using Eqs. (3.4), (3.10), and (2.21) we get

$$\vartheta^2 \sim n \frac{P_2}{\gamma D} L^\gamma, \quad |y|^2 \sim n \frac{\gamma D}{P_2 L^\gamma}. \tag{D14}$$

Substituting the expression (D14) for $|y|^2$ into Eq. (D13), we conclude that $\mu_* v_*$ increases with n. In addition, at $R_- < L$ the Hamiltonian (D11) should be equal to $H_1 \approx H_c$ [Eq. (3.22)], that is,

$$-\mu_* v_* + \frac{\mu^2}{2d} v_* \approx H_c .$$

At $v_* \gg 1$ the relation leads to

$$\mu_* \approx 2d, \quad v_* \sim \sqrt{\gamma n/d}. \tag{D15}$$

Actually, $\mu \approx 2d$ during the whole last substage where $v \gg 1$. Deriving the expression (D15) for v_*, we used the estimates (D13) and (D14).

Now we can turn to the calculation of the effective action (3.13). As before, the main contribution to the action is made by the first stage. Hence

$$i\mathcal{I} = -\frac{1}{\gamma}\int \left(\mu\, dv + \frac{H_1}{G} dv\right). \tag{D16}$$

Moreover, only the contribution related to the vicinity of the reverse point is relevant. Using the relation (3.21) we obtain

$$i\mathcal{I} \approx H_c \ln\frac{L}{r}. \tag{D17}$$

Here we substituted $H_1 \approx H_c$ and took into account that μ has no singular denominator, which is clear from the relation (3.17). Finally, we can determine the structure functions S_n [Eq. (1.2)] in accordance with Eq. (3.2). Collecting Eqs. (D14) and (D17), we obtain the expression (3.33) of the main text.

The expression (D15) shows that indeed $v_* \gg 1$ (which is the applicability condition of the above consideration) at $n \gg n_c$ if there are no additional small parameters. The next subsection is devoted to special situations appearing in the presence of such parameters.

3. Special cases

In this subsection we treat three special cases that are realized at $n > n_c$ if $\gamma \ll 1$, $2-\gamma \ll 1$, or n is extremely large. The general picture given in Appendix D 2 does not essentially change, but particular answers should be slightly corrected.

The case $\gamma \ll 1$ needs special consideration since, as follows from Eq. (D15), the condition $v_* \gg 1$ is violated at $n \sim d/\gamma \gg n_c \sim d\gamma$. Therefore, an intermediate region exists at $n_c \ll n \ll d/\gamma$, where $\gamma \ll v_* \ll 1$. Then we get from Eq. (3.14)

$$H = -\mu v + \frac{\mu^2}{2d} + \frac{|y|^2}{D} R_+^\gamma [\chi(R_+) - \chi(R_-)], \tag{D18}$$

where

$$R_- = R_+ \exp(v/\gamma). \tag{D19}$$

Equations (3.16) now read

$$\frac{dv}{d\xi} = -\gamma v + \gamma \frac{\mu}{d}, \tag{D20}$$

$$\frac{d\mu}{d\xi} = \gamma\mu + \frac{|y|^2}{D} R_+^\gamma R_- \chi'(R_-). \tag{D21}$$

At the end of the first stage $R_- \sim L$. As follows from the relation (D19), due to $v_* \gg \gamma$, the estimate $R_+ \ll L$ is valid. Therefore, the integral (3.4) for ϑ^2 is determined by the time interval between the moments when R_+ and R_- go through L. We can again pass to integrating over R_+ in accordance with Eq. (3.10). Though $R_+ \ll L$ at $R_- \sim L$, $\gamma R_+^\gamma \approx R_-^\gamma (1 - v)$ due to the smallness of γ. Therefore, we return to the estimates (D7). Next we can estimate μ_* from Eq. (D21) where the term $\gamma\mu$ is dropped. Since $R_+^\gamma \sim L^\gamma$ at $R_- \sim L$, integrating over ξ we obtain from Eq. (D21)

$$\mu_* \sim \frac{|y|^2}{D} L^\gamma P_2. \tag{D22}$$

To justify the estimate (D22), one should check that v does not vary essentially in the region where $R_- \sim L$. Using Eq. (D20), we obtain that the condition is satisfied. Substituting formulas (D7) into Eq. (D22), we obtain the previous relation $\mu_* v_* \sim \gamma n \gg |H_c|$. Equating then the Hamiltonian (D18) to H_c at $v = v_*$ we get

$$\mu_* \sim \sqrt{d\gamma n}, \quad v_* \sim \sqrt{\frac{\gamma n}{d}}. \tag{D23}$$

The estimate (D23) for v_* surprisingly coincides with Eq. (D15). Next we obtain from Eq. (D7)

$$\vartheta^2 \sim \sqrt{\frac{n}{n_c}} L^\gamma \frac{n P_2}{D}. \tag{D24}$$

Therefore, the expression (3.33) should be corrected and we get

$$S_n \sim \left(\sqrt{\frac{n}{n_c}} \frac{P_2 C_3}{D} L^\gamma \right)^{n/2} \left(\frac{r}{L}\right)^{\zeta_c}, \tag{D25}$$

where C_3 is a nonuniversal parameter of order unity and ζ_c is defined by Eq. (3.32).

Some peculiarities appear also at $2-\gamma\ll 1$. In this case the function (3.15) can be approximated as

$$\phi\approx 2dv\ln v$$

if $v\gg 1$ but $(2-\gamma)\ln v\ll 1$. Then we obtain from Eq. (3.14)

$$H=-\mu v+\frac{2-\gamma}{2d}v\ln v\,\mu^2+\frac{|y|^2}{D}R_+^{\gamma}[\chi(R_+)-\chi(R_+v^{1/\gamma})]. \tag{D26}$$

The expression (D26) has the same structure as Eq. (D11) except for the logarithmic factor. Taking into account $v\gg 1$, we conclude that the Hamiltonian (D26) leads to the conservation law (D12) which is satisfied with the same accuracy $1/v$ as previously. Therefore, we get instead of Eq. (D13)

$$\mu_*v_*\sim\sqrt{\frac{d|y|^2P_2}{D(2-\gamma)\ln v_*}L^{\gamma}}\sim\sqrt{\frac{nd}{(2-\gamma)\ln v_*}}, \tag{D27}$$

where we used Eq. (D14); its validity accounts for the inequality $v\gg 1$. One can check that the main contribution to H in Eq. (D26) at the end of the first stage is made by $-\mu v$. Equating $-\mu_*v_*$ to H_c we get

$$\ln v_*\sim\frac{n}{n_c}. \tag{D28}$$

Now we obtain the condition $n\ll n_c/(2-\gamma)$, under which the regime under consideration is realized. Since $v\gg 1$ in the regime, we have the same expression (D14) for ϑ and, consequently, the same estimate (3.33) for the structure functions.

We finish this subsection with a discussion concerning extremely large n. In accordance with the estimates (D15), v_* increases with increasing n. Therefore, the last substage of the first stage where $v\gg 1$ starts to play an essential role. First of all we should correct Eq. (3.21), returning to Eq. (3.20). If $v=v_*$ there, then $v_*R_+^{\gamma}$ on the right-hand side of the relation can be substituted by L^{γ}. The contribution to the integral on the left-hand side of Eq. (3.21) associated with the last substage (when v increases from 1 to v_*) can be found by substituting $G(x)\approx -x$ there. Thus the substage makes the additional logarithmic contribution to the integral. Taking the contribution into account, we get instead of Eq. (3.27)

$$\sqrt{\frac{d}{8(2-\gamma)}}\frac{2\pi}{\sqrt{(H_c-H_1)}}=\gamma\ln\frac{L}{r}-2\ln v_*. \tag{D29}$$

There appears also an analogous correction to the expression (D17). Calculating the action (D16) we get

$$i\mathcal{I}\approx H_c\left(\ln\frac{L}{r}-2\frac{\ln v_*}{\gamma}\right). \tag{D30}$$

Here we neglected the term originated from $\int\mu\,dv$ since using Eq. (D15) the term can be estimated as $\sqrt{n/\gamma d}$ and is consequently negligible in comparison to the terms $\sim n$ entering S_n via ϑ. Substituting the expression (D30) into Eq. (3.2), we conclude that the correction related to the last contribution in Eq. (D30) can be neglected in comparison to the strong n dependence of ϑ^n. Nevertheless, the presence of the additional term on the right-hand side of Eq. (D29) shows that our approach is broken at $2\ln v_*\sim\gamma\ln(L/r)$ since H_c-H_1 ceases to be small there. Using the estimates (D15), we arrive at the condition (3.35), which is the applicability condition of our approach. Note that at $\ln(n/d)\gtrsim\gamma\ln(L/r)$ the scaling behavior of S_n is destroyed.

4. Instanton near critical order

Here we will consider the case when n is close to the critical value (3.28): $|n-n_c|\ll n_c$. Then $v_*\ll 1$ and upon examining the last substage of the first stage and of the second stage we can use the Hamiltonian, expanded over small v.

If $n<n_c$ then we can take the Hamiltonian (D1). The only peculiarity of the consideration near n_c is that the logarithmic contributions to the effective action (3.13) and to the left-hand side of the integral (3.20) depend on $n-n_c$. Indeed, the expression (3.25) shows that $G(x)$ is a linear function of x only in the restricted interval

$$\frac{2-\gamma}{2d\gamma}|H_c-H_1|>v>v_*.$$

Just the interval determines the value of the logarithm. This means that we should substitute in the expressions (D9) and (D10)

$$\ln\frac{1}{v_*}\to\ln\frac{v_*n_c^2}{(n-n_c)^2}.$$

The substitution does not change the final answer (D10) for the action but influences the value of ϑ^2 because of Eq. (D7). Taking the fact into account we get the expression (3.36) that replace Eq. (3.29).

It is clear that at $n\to n_c$ the difference $n-n_c$ starts to compete with r/L, which destroys our construction. Let us estimate the corresponding value of n_c-n regarding $n<n_c$. It follows from the asymptote (3.25) that our consideration is correct if $(H_c-H_1)\gg d\gamma v_*/(2-\gamma)$. Expressing in the inequality v_* via r/L from Eq. (3.24) we obtain the criterion (3.37). Note that the criterion implies the inequality

$$\gamma\ln\frac{L}{r}\gg 1,$$

which should be assumed for our treatment to be correct. Otherwise special consideration is needed.

Let us now proceed to the case $n>n_c$. We should take into account the next term of the expansion of H over v in comparison with Eq. (D1), which should correct the value (D6) of H_1 since we know that H_1 is equal to H_c. The corresponding expression is

$$H=-\mu v+\frac{2(2-\gamma)}{d\gamma^2}\left(1-\frac{4-\gamma}{2\gamma}v\right)\mu^2v^2+\frac{|y|^2}{D}R_+^{\gamma}[\chi(R_+)-\chi(R_-)], \quad (D31)$$

where the difference $\chi(R_+)-\chi(R_-)$ should be expanded up to second order over v. Equations (3.16) with the Hamiltonian (D31) lead to

$$\frac{1}{\gamma}\frac{dv}{d\xi}=-v+\mu v^2(\alpha_1-\alpha_2 v), \quad (D32)$$

$$\frac{1}{\gamma}\frac{d}{d\xi}(\mu v)=\frac{1}{2}\alpha_2\mu^2v^3+\frac{|y|^2}{D}R_+^{\gamma+1}\chi'(R_-)v, \quad (D33)$$

$$\alpha_1=\frac{4(2-\gamma)}{d\gamma^2},\quad \alpha_2=\frac{2(4-\gamma)(2-\gamma)}{d\gamma^2}. \quad (D34)$$

Equating the Hamiltonian (D31) to the value (3.22), we get

$$\mu v\approx\frac{\gamma n_c}{2}, \quad (D35)$$

at the end of the first stage. Note that due to $v_*\ll 1$, we can use the estimates (D7).

Our next aim is to estimate v_* in terms of $n-n_c$. To do so we will use the identity $dH/d\xi=\partial H/\partial\xi$ that is correct for any canonical system. Let us integrate the relation over the second stage. Since $H=H_1$ at the first stage and $H=0$ at the end of the second stage we obtain

$$-H_1=\frac{|y|^2}{D}\int d\xi\{\gamma R_+^{\gamma}[\chi(R_+)-\chi(R_-)]+R_+^{\gamma}[R_+\chi'(R_+)-R_-\chi'(R_-)]\}. \quad (D36)$$

Expanding the right-hand side of the relation (D36) up to second order over v, integrating by parts (to remove high derivatives of χ), and using Eqs. (3.12), (D32), (D33), and (2.21), one can obtain the expression

$$-H_1=\frac{\gamma n}{2}\left(1-\frac{\gamma\phi_1 n}{4}\right)-\frac{\gamma\alpha_1^2}{2}(n-n_c)\frac{|y|^2}{D}\times\int d\xi R_+^{1+\gamma}\chi'(R_+)\mu v^3-\frac{|y|^2\alpha_2}{D}\int d\xi R_+^{1+\gamma}\chi'(R_+)\mu v^3-\frac{\gamma|y|^2}{2D}\alpha_1\alpha_2\int d\xi R_+^{1+\gamma}\chi'(R_+)v\int_{\xi}^{\infty}d\tilde{\xi}\,\mu^2v^3. \quad (D37)$$

The first term on the right-hand side of Eq. (D37) reproduces the previous result (D6) and the other terms are small corrections proportional to v_*.

We know that in the main approximation over r/L we can substitute $H_1=H_c$. Regarding $n-n_c\ll n_c$, we neglect the second term on the right-hand side of Eq. (D37). The last two terms on the right-hand side of Eq. (D37) can be estimated using $\mu_*v_*\sim\gamma n_c$, $\chi\sim P_2$, $R_+\sim L$, and the estimation (D7) for $|y|^2$. Combining all together we get

$$v_*\sim\gamma\frac{(n-n_c)^2}{n_c^2}. \quad (D38)$$

Let us stress that due to $\alpha_2>0$ the last two terms on the right-hand side of Eq. (D37) are positive if $\chi(R)$ is a monotonic function, which is a reasonable condition. Therefore, a solution of Eq. (D37) for v_*, determined by the estimate (D38), really exists. We conclude that there is an instantonic solution with nonmonotonic behavior of v at any $n-n_c\ll n_c$.

The last assertion should be corrected since it is true only if $L/r\to\infty$. At finite L/r there is a narrow region of very small $n-n_c$ where our scheme does not work. To establish the corresponding criterion let us recall that our consideration is valid if $v_r\ll v_*$. It follows from Eq. (3.27) that $v_r\sim 1/\gamma\ln^2(L/r)$. Then, from the formula (D38) we get the inequality (3.37).

Finally, we can determine the structure functions S_n [Eq. (1.2)] in accordance with Eq. (3.2). Collecting Eqs. (D7), (D17), and (D38), we obtain the formula (3.36).

[1] A. N. Kolmogorov, C. R. Acad. Sci. URSS **30**, 301 (1941).
[2] H. L. Grant, R. W. Stewart, and A. Moilliet, J. Fluid Mech. **12**, 241 (1961).
[3] G. K. Batchelor and A. A. Townsend, Proc. R. Soc. London, Ser. A **199**, 82 (1962).
[4] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
[5] G. K. Batchelor, *Theory of Homogeneous Turbulence* (Cambridge University Press, New York, 1953).
[6] A. Monin and A. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975).
[7] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, New York, 1995).
[8] A. M. Obukhov, Izv. Acad. Nauk. SSSR, Geogr. Geofiz. **13**, 58 (1949).
[9] S. Corrsin, J. Appl. Phys. **22**, 469 (1951).
[10] R. Antonia, E. Hopfinger, Y. Gagne, and F. Anselmet, Phys. Rev. A **30**, 2704 (1984).
[11] C. Meneveau and K. R. Sreenivasan, Phys. Rev. A **41**, 2246 (1990).
[12] K. R. Sreenivasan, Proc. R. Soc. London, Ser. A **434**, 165 (1991).
[13] I. Hosokawa and K. Yamamoto, in *Turbulence and Coherent Structures*, edited by O. Metais and M. Lesieur (Kluwer, London, 1991).
[14] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[15] B. Shraiman and E. Siggia, C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron. **321**, 279 (1995).
[16] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).
[17] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[18] M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1995).
[19] D. Bernard, K. Gawedzki, and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996).
[20] B. Shraiman and E. Siggia, Phys. Rev. Lett. **77**, 2463 (1996).
[21] D. Gutman and E. Balkovsky, Phys. Rev. E **54**, 4435 (1996).
[22] E. Balkovsky, G. Falkovich, and V. Lebedev, Phys. Rev. E **55**, R4881 (1997).
[23] R. H. Kraichnan, V. Yakhot, and S. Chen, Phys. Rev. Lett. **75**, 240 (1995).
[24] A. L. Fairhall, B. Galanti, V. S. L'vov, and I. Procaccia, Phys. Rev. Lett. **79**, 4166 (1997).
[25] U. Frisch, A. Mazzino, and M. Vergassola, Phys. Rev. Lett. **80**, 5532 (1998).
[26] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[27] V. Yakhot, Phys. Rev. E **55**, 329 (1997).
[28] M. Chertkov, Phys. Rev. E **55**, 2722 (1997).
[29] P. C. Martin, E. Siggia, and H. Rose, Phys. Rev. A **8**, 423 (1973).
[30] C. de Dominicis, J. Phys. (Paris), Colloq. **37**, C1-247 (1976).
[31] H. Janssen, Z. Phys. B **23**, 377 (1976).
[32] G. Falkovich, I. Kolokolov, V. Lebedev, and A. Migdal, Phys. Rev. E **54**, 4896 (1996).
[33] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).
[34] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **78**, 1452 (1997).
[35] G. Falkovich and V. Lebedev, Phys. Rev. Lett. **79**, 4159 (1997).
[36] E. Balkovsky and G. Falkovich, Phys. Rev. E **57**, 1231 (1998).
[37] B. Shraiman and E. Siggia, Phys. Rev. E **49**, 2912 (1994).
[38] L. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[39] A. N. Kolmogorov, J. Fluid Mech. **12**, 82 (1962).
[40] M. Lesieur, *Turbulence in Fluids* (Kluwer Academic, Dordrecht, 1997).
[41] D. Bernard, K. Gawedzki, and A. Kupiainen, J. Stat. Phys. **90**, 519 (1998).
[42] U. Frisch, A. Mazzino, and M. Vergassola (unpublished).
[43] R. H. Kraichnan, Phys. Fluids **9**, 1937 (1966).