Universal direct cascade in two-dimensional turbulence

Gregory Falkovich and Vladimir Lebedev*
Physics Department, Weizmann Institute of Science, Rehovot 76100, Israel
(Received 31 May 1994)

We consider the correlation functions of vorticity ω in the region of the direct cascade in a steady two-dimensional turbulence. The nonlocality of the cascade in k space provides for logarithmic corrections to the expressions obtained by dimension estimates, and the main problem is to take those logarithms into account. Our procedure starts directly with the Euler equation rewritten in the comoving reference frame. We express the correlation functions of the vorticity via the correlation functions of the pumping force and renormalized strain. It enables us to establish a set of integrodifferential equations which gives a logarithmic renormalization of the vorticity correlation functions in the inertial interval. We find the indices characterizing the logarithmic behavior of different correlation functions. For example, the two-point simultaneous functions are as follows: $\langle\omega^n(\mathbf{r}_1)\omega^n(\mathbf{r}_2)\rangle \sim [P_2 \ln(L/\mid \mathbf{r}_1-\mathbf{r}_2\mid)]^{2n/3}$, where L is the pumping scale. We demonstrate that the form of those correlation functions is universal, i.e., independent of the pumping. The only pumping-related value which enters the expressions is the enstrophy production rate P_2. The contributions related to pumping rates P_n of the higher-order integrals of motion are demonstrated to be small in comparison with the ones induced by P_2. We establish also the time dependence of the correlation functions, the correlation time τ in the comoving reference frame is the same for the vorticity and strain and is scale dependent: $\tau \propto \ln^{2/3}(L/r)$. We reformulate our procedure in the diagrammatic language to reinforce the conclusions.

PACS number(s): 47.10.+g, 47.27.Gs

INTRODUCTION

Remarkable feature of incompressible fluid turbulence is the presence of an essential interaction between eddies of strongly different scales due to a sweeping of small eddies by large ones. Such interaction manifests itself differently in two and in three dimensions. In the three-dimensional case, it was shown that sweeping has strong effect upon the time dependencies of the correlation functions while the simultaneous velocity correlators are determined by a dynamical interaction which is local in k space [1–3]. Due to simple geometrical reasons, the sweeping has stronger consequences in two rather than in three dimensions. Even if one considers simultaneous correlators, logarithmic infrared divergences are present for the spectrum $E(k) \propto k^{-3}$ obtained for a vorticity cascade from a dimensional analysis [4]. The presence of the divergences means that the nonlocal interaction should play a substantial role in shaping the energy spectrum. Since the powers of the logarithm increase with the order of perturbation theory, then a renormalization of the spectrum might occur. All the more difficult, though necessary to formulate a consistent theory, is to study time correlations. By presuming weak time correlations and using a one-loop approximation, Kraichnan found the spectrum $E_k \sim k^{-3}\ln^{-1/3}(kL)$ [4]. This estimate can be obtained also by different uncontrollable closures assuming weak phase coherence (see [5] and references therein). A natural question arises: does the account of higher orders and of (at least substantial) time correlations destroy this spectrum.

After summation, those logarithms could, in principle, change the exponent -3 in the energy spectrum. The alternative exponents -4 and $-11/3$ suggested by Saffman [6] and Moffatt [7] cannot thus be rejected from the general point of view. Moreover, there has been even formulated the viewpoint that the small-scale asymptotics are not universal in two-dimensional (2D) turbulence so that one encounters different scaling laws under the different conditions of excitation [8]. This possibility may be associated with the existence of an infinite number of integrals of motion (which are powers of the vorticity ω) in the 2D inviscid dynamics. By varying the pumping statistics, one changes the inputs of different integrals which could drastically change the character of the leading contributions to the correlation functions of ω. For example, such a nonuniversality would take place if the direct 2D cascade is described by the theory of conformal turbulence [9]. The condition of conformal invariance is so restrictive in 2D that, for a given conformal model, it prescribes scaling exponents as well as numerical prefactors for all correlation functions, the choice of the model being related to the peculiarities of the pumping.

Here, we prove that a steady direct cascade is universal in two dimensions, in contrast to those expectations. Our conclusions are based on the formalism starting directly from the Euler equation and enabling us to express any correlation function of ω via the pumping. We see that all simultaneous correlation functions tend to uni-

*On leave from Landau Institute for Theoretical Physics, Moscow, GSP-1, Kosygina 2, Russia.

versal small-scale limits that depend only on one number determined by the pumping: the input rate P_2 of the enstrophy (squared vorticity).

Our theory is a consistent formalization of the generally accepted physical picture of the vorticity cascade: a fluid blob embedded into a larger-scale velocity shear is extended along the direction of a positive strain value and compressed along the direction of a negative one; such stretching provides for the vorticity flux into the small scales with the rate of transfer proportional to the strain. Vorticity rotates the blob decelerating stretching due to interchange of the axis of a positive and negative strain. We show that vorticity correlators indeed are solely determined by the influence of larger scales, which can be described in terms of the tensor of velocity derivatives with a symmetric part (strain) and an antisymmetric one (vorticity). The problem of determining vorticity spectrum in the inertial interval turns thus to be the problem of passive scalar [11–13] advected by the velocity field produced by the previous (larger) scales.

As one passes into smaller scales, the effective vorticity and strain that act on the scalar are renormalized. To find the law of renormalization one should take into account time correlations between the velocity gradients produced by different spectral intervals. The sketch of the present theory has been published before [10] where we found, in particular, that the form of the pair correlator is the same whether you presume a short correlation time of the renormalized strain or a long one. This is associated with the fact that the renormalized interaction vertex is of the order of its bare value: not only a power-like but even a logarithmic renormalization of the vertex is absent. Recent numerics confirm the prediction on the spectrum [14]. The true spectrum thus coincides with Kraichnan's prediction although the approximation of a short correlation time is not correct. Here, we demonstrate that the universal regime corresponds to an opposite "slow" case: correlation times are logarithmically large in the inertial interval (in the locally comoving reference frame). Because of a simple stretching nature of spectral transfer, the typical strain correlation time will be shown below to be of the order of the time of spectral transfer. Note that the remarkable persistence of the local straining has been experimentally observed by Townsend more than forty years ago [15] and was not theoretically proved until now. We demonstrate that by a direct calculation of the correlation times of the strain and vorticity.

Our problem thus turns out to be that of a scalar (vorticity) advected by a slow velocity field (with the correlation time much larger than the turnover time). That problem is shown here to be solvable in 2D: one can write any vorticity correlation function expressed via the average of the known function of the strain and vorticity itself. Those expressions form an infinite set of integro-differential equations for the correlation functions. We cannot solve the equations directly (i.e., we cannot find the correlation functions exactly with a numerical factor) yet we can establish the homogeneity property of the probability distribution function that should satisfy the set of the equations. The self-similar solution is logarithmic and we can explicitly write in the main logarithmic approximation the expression for any correlation function up to a dimensionless numerical factor. We obtain the logarithmic renormalization of the strain and vorticity, which makes the procedure self-consistent since it actually enables one to neglect the interaction of modes with comparable scales in comparison with the stretching by a larger-scale velocity field. Note that our conclusions are obtained, strictly speaking, only for a steady turbulence with vorticity pumping.

It is well-known that two-dimensional turbulence may contain isolated long-living vortices [16]. We show that the main contribution into the small-scale tails of the correlation functions is provided by the low-vorticity regions between the vortices and the form of the correlation functions is insensible to the presence of large-scale isolated vortical structures.

The structure of the paper is as follows. We formulate the problem in Sec. I. In Sec. II, we consider an auxiliary (yet important by itself) problem of passive scalar advection by an external large-scale velocity field, paying the main attention to the case of a slow velocity. In Sec. III, we find renormalization law of the stretching rate for the direct cascade utilizing the same ideas as for the passive scalar problem. It enables us to establish the character of even correlation functions of vorticity. In Sec. IV, we consider the problem of an infinite number of integrals of motion in 2D turbulence and the related problem of the structure of the odd correlation functions of ω. We introduce the notion of a "distributed" pumping and explain why only the enstrophy flux is constant in the inertial interval of scales. In Conclusion, we shortly enumerate the results and applicability conditions of our theory and discuss how universality appears at small scales. Our theory does not imply that other (nonlogarithmic) solutions cannot exist. However, even if such spectra could be matched with some sources of special form (which is yet unclear) they should be structurally unstable with respect to pumping variations that produce logarithmic tail which provides for the main contribution at small scales. There are also two Appendices devoted to some technical details. In Appendix A, we give the diagrammatic justification of the procedure developed in Secs. III and IV. In Appendix B, we formulate some integral relations enabling us to transfer the vorticity correlation functions into the strain correlation functions and to prove that the correlation times of the vorticity and of the strain coincide.

I. FORMULATION OF THE PROBLEM

We consider the structure of the correlation functions of the 2D turbulent vorticity $\omega = \mathrm{curl}\,\mathbf{u}$ in the inertial interval of scales r determined by the inequalities $L_{vis} \ll r \ll L$, where L is the characteristic length of the pumping force and L_{vis} is the viscous length. This is just the interval of scales where the direct cascade occurs and where it is possible to neglect the viscous term in the Navier-Stokes equation. We consider the Euler equation containing the random external force $\phi(t,\mathbf{r})$ acting on the vorticity:

$$\partial_t\omega + u_\alpha\nabla_\alpha\omega = \phi\,, \tag{1.1}$$

where $\partial_t \equiv \partial/\partial t$. Our aim is to express small-scale asymptotics of the vorticity correlation functions in a steady turbulent state via the correlation functions of ϕ. To eliminate homogeneous sweeping, we pass to the locally comoving reference frame introducing the quasi-Lagrangian (qL) velocity $\mathbf{v}(t,\mathbf{r})$ related to the Eulerian velocity $\mathbf{u}$ as

$$\mathbf{u}(t,\mathbf{r}) = \mathbf{v}\left(t,\mathbf{r} - \int^t dt'\,\mathbf{v}(t',0)\right)\,. \tag{1.2}$$

The presence of a marked point $\mathbf{r}=\mathbf{0}$ makes the theory spatially nonuniform in qL variables [2,3]. We do not know of a way to pay a lower price for sweeping elimination. The equation (1.1) in qL variables takes the form

$$\partial_t\omega + (v_\alpha - v_{0\alpha})\nabla_\alpha\omega = \phi\,, \tag{1.3}$$

where $v_{0\alpha} = v_\alpha(t,0)$. Note that simultaneous correlators are the same for both sets of variables.

The source ϕ can be assumed to be δ-correlated in time in the qL variables. The point is that the turbulent velocity usually contains scales larger than L so that the mean turbulent velocity V_0 is determined by the largest scale while the strain and the vorticity are determined by the eddies with the scale L. This means that the typical correlation time of ϕ in the comoving frame is L/V_0, which is much less than the turnover time of L eddies whatever be the correlation time of ϕ in the laboratory frame. Physically this corresponds to accounting for the existence of the inverse cascade which is unavoidable in a consistent theory of the direct cascade [17]: if the inverse cascade was absent (say, by pumping the largest scale in the system) then the direct cascade might be different from what is discussed in this paper.

II. PASSIVE SCALAR STRETCHING

As a first step, we consider an advection of the passive scalar θ by a long-range velocity. In addition to being a paradigm for the further consideration of the vorticity cascade, this problem is interesting by itself. Examples of θ are the temperature field or the concentration of impurities in the fluid. Since the velocity is assumed to be independent of the scalar then there are two different characteristic scales in the problem: the length L_1 determines the characteristic size of "heaters" that are the sources of the passive scalar and the length L_2 determines the smallest scale of the advecting velocity. The formalism developed in this section is correct for small-scale asymptotics of the correlation functions of the passive scalar θ realized at scales $r \ll L$, where $L = \min(L_1, L_2)$. Since we will neglect the diffusion of the passive scalar θ the applicability condition of our theory is $L_{dif} \ll L$, where L_{dif} is the diffusive length. This situation is realized, e.g., in the viscous-convective range at large Prandtl number Pr (viscosity to diffusivity ratio), where the role of L_2 is played by the viscous length L_{vis}. The general theory of random advection in two dimensions has been developed in [13], here we briefly review the principal ideas of our approach. The main attention will be paid to the case of a slow advecting velocity since just this case is closely related to the problem of vorticity cascade.

The dynamic equation for the passive scalar θ formally coincides with the Eq. (1.1), where instead of the vorticity ω (which is a scalar in 2D), we take θ and treat the velocity as a long-range one, this means that the velocity contains only wave vectors $q \lesssim L^{-1}$. We will designate this velocity by $\mathbf{V}$. After passing to qL variables, we get the equation

$$\partial_t\theta + (V_\alpha - V_{0\alpha})\nabla_\alpha\theta = \phi\,, \tag{2.1}$$

analogous to (1.3). Further, we proceed in the spirit of Kraichnan's approach [12]. For the points with $|\,\mathbf{r}\,| < L$, one can expand $V_\alpha(t,\mathbf{r}) - V_\alpha(t,0)$ in the series over $\mathbf{r}$, the first nonvanishing term of this expansion is $\sigma_{\alpha\beta}(t)r_\beta$, where $\sigma_{\alpha\beta}(t) = \nabla_\beta V_\alpha$ at $\mathbf{r}=0$. Keeping only this term, we find

$$\partial_t\theta(t,\mathbf{r}) + \sigma_{\alpha\beta}(t)r_\beta\nabla_\alpha\theta(t,\mathbf{r}) = \phi(t,\mathbf{r})\,. \tag{2.2}$$

A solution of this equation can be represented in the following form

$$\theta(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\,\phi(t',\hat{w}(t,t')\mathbf{r})\,, \tag{2.3}$$

where the matrix $\hat{w}$ is an antichronological exponent [12,10,13] which satisfies the equation

$$\partial_{t'}\hat{w}(t,t') = \hat{\sigma}(t')\hat{w}(t,t')\,, \tag{2.4}$$

with the initial condition $\hat{w}(t,t)=1$. Due to incompressibility, the matrix $\hat{\sigma}$ is traceless so it can be expressed via three scalar functions: $\sigma_{xx} = -\sigma_{yy} = a$, $\sigma_{xy} = b + c$, and $\sigma_{yx} = b - c$. The independent random processes $a(t), b(t)$, and $c(t)$ are stationary with zero mean. Respective correlation functions of a and b coincide due to isotropy.

First, we examine the expression for the correlation functions following from (2.3) to recognize what kind of information we should extract from (2.4). By using (2.3), the correlation functions of the scalar θ in the locally comoving reference frame can be rewritten in terms of the given correlation functions of the pumping integrated along the fluid path. The source ϕ can be assumed to be δ correlated in time in the qL variables, the reasons for this were discussed in the Introduction. We thus write

$$\langle\phi(\mathbf{r}_1,t_1)\phi(\mathbf{r}_2,t_2)\rangle = P_2\chi(|\mathbf{r}_1 - \mathbf{r}_2|)\delta(t_1 - t_2)\,, \tag{2.5}$$

where the function $\chi(r)$ describes spatial correlations of the pumping. We put $\chi(0)=1$ then the constant P_2 in (2.5) is the production rate of a squared scalar. The expressions analogous to (2.5) can be written also for higher-order correlation functions of the force ϕ, those functions being related to the pumping rates of higher-order integrals of motion. The simultaneous pair correlation function of θ can be found from (2.3)

$$\langle\theta(t_1,\mathbf{r}_1)\theta(t_1,\mathbf{r}_2)\rangle = P_2\left\langle\int_0^\infty d\varsigma\,\chi(|\,\boldsymbol{\rho}(t_1,t_1-\varsigma)\,|)\right\rangle_\sigma , \tag{2.6}$$

where $\boldsymbol{\rho}(t_1,t) = \hat{w}(t_1,t)(\mathbf{r}_1-\mathbf{r}_2)$ and $\langle\ldots\rangle_\sigma$ denotes the average over the statistics of $\hat{\sigma}$ that is over the statistics of a,b,c. Averaging with respect to the statistics of both the external velocity gradients $\hat{\sigma}$ and of the external source ϕ is implied in the left-hand side of (2.6).

At calculating (2.6), we can put simply $\chi(x)=1$ for $x<L$ and $\chi(x)=0$ for $x>L$ since the account of any given shape of $\chi(x)$ will give the same results with the logarithmic accuracy. Then

$$F_2 = \langle\theta(t_1,\mathbf{r}_1)\theta(t_1,\mathbf{r}_2)\rangle = P_2\tau_*(r_{12}) , \tag{2.7}$$

where τ_* is an average value of time ς which is necessary for ρ to grow from $r_{12}=|\,\mathbf{r}_1-\mathbf{r}_2\,|$ to L at increasing ς in (2.6). To find the time τ_*, which can be called the spectral transfer time, one should solve the equation

$$\partial_\varsigma\rho_\alpha(t_1,t_1-\varsigma) = -\sigma_{\alpha\beta}\rho_\beta , \tag{2.8}$$

following from (2.4): the initial condition for this equation is $\boldsymbol{\rho}(t_1,t_1)=\mathbf{r}_1-\mathbf{r}_2$. The equation (2.8) can be rewritten in the polar coordinates $\boldsymbol{\rho}=(\rho\cos\vartheta,\rho\sin\vartheta)$:

$$\partial_\varsigma\rho = \alpha\rho , \qquad \partial_\varsigma\vartheta = \beta + c , \tag{2.9}$$

where

$$\alpha = -a\cos(2\vartheta) - b\sin(2\vartheta), \quad \beta = a\sin(2\vartheta) - b\cos(2\vartheta) .$$

It is remarkable that the equation for ϑ is separated, it can be treated as a constrain enabling us to express the angle ϑ via the fields a,b. After that is done, the equation (2.9) for ρ becomes a scalar equation with the solution

$$\ln[\rho(t_1,t)/r] = \int_t^{t_1} dt'\,\alpha(t') . \tag{2.10}$$

Thus, we have reduced our matrix problem to the scalar one. Since the right-hand side (r.h.s.) of (2.10) is an integral over time then the statistics of $\ln(\rho/r)$ is asymptotically Gaussian for t_1-t much larger than the correlation time of the random process $\alpha(t')$. Note that the same procedure with the same conclusion on asymptotic local Gaussianity is correct for the stretching problem in the space of arbitrary dimension d. The complete analysis of the solution of the equations (2.2–2.10) is published elsewhere [13]. It is demonstrated, in particular, that for quite an arbitrary random velocity field, the mean stretching rate (Lyapunov exponent),

$$\bar{\lambda} = \lim_{t\to\infty}\int_0^t \alpha(t')dt'/t,$$

is positive. That means that the stretching is exponential so that the stretching time τ_* depends on the distance r_{12} logarithmically as well as the correlation function — see [13] and (2.14) below. Here, we describe in more detail the particular case of the slow velocity field since it will arise in the next sections at the consideration of the nonlinear problem. The notion of the slow-velocity field implies that the correlation time τ_0 of $\hat{\sigma}$ is much larger than the characteristic value of a^{-1},b^{-1},c^{-1}.

For a slow $\hat{\sigma}$, there are two different regimes described by the equation (2.9). The first one is realized if $a^2+b^2>c^2$. In this case, the r.h.s. of (2.9) has two fixed points for ϑ, only one of which is stable. The value of ϑ reaches this point for the time of the order of $(a^2+b^2)^{-1}$ and then adiabatically adjusts to the fixed point slowly displacing at variations of a,b,c. The value of the stretching rate α corresponding to this point is $\sqrt{a^2+b^2-c^2}$ and, therefore, the contribution to the r.h.s. of (2.10) due to a region $a^2+b^2>c^2$ is $\int dt'\sqrt{a^2+b^2-c^2}$. In the opposite case $a^2+b^2<c^2$ the solution of (2.9) has a nearly periodic character with the period $T=2\pi/\sqrt{c^2-a^2-b^2}$, which varies slowly with time. Then α will be also a nearly periodic function. The contribution to the r.h.s of (2.10) is associated with deviations of $\langle\alpha\rangle$ from zero which are small in T/τ_0. The inequality $T\ll\tau_0$ is violated only near a point where $a^2+b^2=c^2$, the vicinity of these points will produce a contribution to the r.h.s. of (2.10) of order unity which can be neglected with our logarithmic accuracy. Therefore, we conclude that

$$\ln[\rho(t_1,t)/r] = \int_t^{t_1} dt'\,\sqrt{a^2(t')+b^2(t')-c^2(t')}, \tag{2.11}$$

where the integration is taken only over the regions where $a^2+b^2>c^2$.

This result can be obtained also in another language. Differentiating (2.8) and neglecting $\partial_\varsigma\sigma$ comparing to σ^2, one obtains the equation $d^2\boldsymbol{\rho}/d\varsigma^2+\hat{\sigma}^2\boldsymbol{\rho}=0$. And here a little miracle happens in 2D: Due to incompressibility, $\hat{\sigma}^2$ is proportional to a unit matrix so that we reduce the matrix equation to a scalar one which can be written in the form

$$\partial_\varsigma^2(\rho_x+i\rho_y) = (a^2+b^2-c^2)(\rho_x+i\rho_y) . \tag{2.12}$$

One can consider (2.12) as a Schrödinger equation for a particle in a random potential $U=a^2+b^2-c^2$; the variable ς plays the role of coordinate. The statement about an exponential stretching is a direct analog of the statement about the localization in a random one-dimensional potential. Our limit of a slow strain corresponds to a quasiclassical regime so that $\boldsymbol{\rho}(\varsigma)$ can be calculated by using semiclassical approximation. Classically forbidden and allowed regions should be considered separately, which corresponds to the two regimes examined above. In the classically allowed regions, $U<0$ and the wave function oscillates while in the classically forbidden regions $U>0$ and the wave function grows exponentially. The rate of the amplitude growth can be found using only exponential factors associated with classically forbidden regions, which immediately gives (2.11). Note that this expression is invalid for $\rho_x+i\rho_y$ which corresponds to a bound state (or is close to it) when ρ grows and then decays as ς increases. The probability of such a coincidence is small and consequently (2.11) can be used at the statistical treatment (for the rigorous proof, see, [13]).

The physical meaning of the above analysis is quite clear. The value $U = a^2 + b^2 - c^2$ is equal to the squared strain minus the squared vorticity. The distinction between the hyperbolic regions ($U > 0$) with a predominant strain and the elliptic regions ($U < 0$) where rotation dominates has been introduced by Weiss [18]. It is clear that long-living vortices have $U < 0$ and give no contribution into the stretching rate.

Since the r.h.s. of (2.11) is an integral over time then it is a self-averaging quantity which at $t_1 - t \to \infty$ has a sharp maximum near $(t_1 - t)\bar{\lambda}$, where

$$\bar{\lambda} = \mathrm{Re}\langle\sqrt{a^2 + b^2 - c^2}\rangle \ . \tag{2.13}$$

Therefore, the value of $\ln(\rho/r)$ determined by (2.11) can be estimated as $\bar{\lambda}(t_1 - t)$, which means that the time τ_* in (2.7) is equal to $\bar{\lambda}^{-1}\ln(L/r_{12})$. Thus, we come to the Batchelor-Kraichnan expression for the simultaneous pair correlation function

$$F_2 = \langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle = \frac{P_2}{\bar{\lambda}}\ln\frac{L}{|\,\mathbf{r}_1 - \mathbf{r}_2\,|} \ . \tag{2.14}$$

For (2.14) to be correct, the time $\tau_* = \bar{\lambda}^{-1}\ln(L/r_{12})$ of the spectral transfer should be larger than the correlation time τ_0 of $\hat{\sigma}$, which is true for sufficiently large values of $\ln(L/r_{12})$. The correlator (2.14) corresponds to $F_2(k) = P_2/\bar{\lambda}k^2$ in the $\mathbf{k}$ space, this gives $E(k) \propto k^{-3}$.

The nonsimultaneous correlation functions of θ also can be obtained from (2.3). Using the expression (2.5), we find that the pair correlation function is

$$\langle\theta_1\theta_2\rangle = P_2\int_{-\infty}^{t} dt'\langle\chi[\hat{w}(t_1,t')\mathbf{r}_1 - \hat{w}(t_2,t')\mathbf{r}_2]\rangle_\sigma \ , \tag{2.15}$$

where $t = \min(t_1, t_2)$. As above, the value of the correlation function is determined by the characteristic time which is needed for the argument of χ in (2.15) to increase to L. For sufficiently small r_1 and r_2, when this characteristic time is larger than τ_0, the absolute values of $\hat{w}(t_1,t')\mathbf{r}_1$ and $\hat{w}(t_2,t')\mathbf{r}_2$ in (2.15) can be approximated by $\exp[\bar{\lambda}(t_1 - t')]r_1$ and $\exp[\bar{\lambda}(t_2 - t')]r_2$, respectively. If $r_1 \sim r_2$ then the argument of χ in (2.15) will be determined by the largest of two times t_1 and t_2, say, t_1 (then $t = t_2$). In this case, the integration over t' in (2.15) is within the interval $t_1 - \bar{\lambda}^{-1}\ln(L/r_1) < t' < t_2$. Therefore, with the logarithmic accuracy

$$\langle\theta(t_1,\mathbf{r}_1)\theta(t_2,\mathbf{r}_2)\rangle = P_2\left[\bar{\lambda}^{-1}\ln(L/r_1) - (t_1 - t_2)\right] , \tag{2.16}$$

which implies $t_1 > t_2$, $r_1 \sim r_2$. The expression (2.16) is correct if $\bar{\lambda}^{-1}\ln(L/r_1) > (t_1 - t_2) > \bar{\lambda}^{-1}, \tau_0$. Note that the correlation function (2.16) does not depend only on the difference $\mathbf{r}_1 - \mathbf{r}_2$ since we lost the homogeneity at passing to the comoving reference frame. We see that the correlation time τ of the passive scalar θ in this frame is logarithmically large independently of the correlation time τ_0 of $\hat{\sigma}$, namely, $\tau = \bar{\lambda}^{-1}\ln(L/r)$. Note that the field of the passive scalar change substantially at a given spatial point during the typical turnover time, i.e., $\bar{\lambda}^{-1}$ while in the frame comoving with the fluid blob that field is long correlated at small scales.

Many-point correlation functions of θ can be extracted from the same relation (2.3). For example, the four-point correlation function is as follows

$$\langle\theta_1\theta_2\theta_3\theta_4\rangle = \int^{t_1} dt_1'\int^{t_2} dt_2'\int^{t_3} dt_3'\int^{t_4} dt_4'\langle\phi_1\phi_2\phi_3\phi_4\rangle \, . \tag{2.17}$$

The reducible part of $\langle\phi_1\phi_2\phi_3\phi_4\rangle$ gives the contribution to $\langle\theta_1\theta_2\theta_3\theta_4\rangle$, which is a product of the pair correlation functions (2.16). In the contribution supplied by the irreducible part of $\langle\phi_1\phi_2\phi_3\phi_4\rangle$, there will be only one integration giving a logarithmic factor so that in the convective interval it is small in comparison with the product of the pair correlation functions proportional to the squared logarithm. The same is true for many-point correlation functions of the order $n < \tau_*/\tau_0$: the main contribution to the correlation functions is supplied by their reducible parts. It means that for a given convective interval determined by the Prandtl number Pr the Gaussianity is correct up to the number $n \simeq \ln \mathrm{Pr}/(\bar{\lambda}\tau_0)$.

We conclude that for large enough Pr the local statistics of the passive scalar θ advected by a large-scale velocity field appears to be asymptotically Gaussian irrespective of the statistics of the external influence ϕ and of the statistics of the advecting velocity V_α. The rigorous proof of this statement and the analysis of the non-Gaussian tails of the probability distribution at finite values of τ_*/τ_0 and $\ln\mathrm{Pr}$ can be found in [13]. To avoid a misunderstanding, note that the asymptotic Gaussianity of the passive scalar is established by only temporal averaging in the locally comoving reference frame. If there are separate space regions with different values of the pumping (the flow is nonergodic) and if one desired to average with respect to such a superensemble, then Gaussianity is lost while the logarithmic dependencies of the correlation functions persist.

III. STRAIN RENORMALIZATION

Now let us turn to the nonlinear problem of describing vorticity cascade. The velocity $\mathbf{v}$ is now connected to the vorticity ω:

$$v_\alpha(\mathbf{r}) = \epsilon_{\alpha\gamma}\nabla_\gamma\int d^2r'\omega(\mathbf{r}')\ln(L/R)/2\pi, \tag{3.1}$$

with $R = |\mathbf{r}' - \mathbf{r}|$. Therefore, the velocity $\mathbf{v}$ contains harmonics of all scales from L to the viscous scale L_{vis} and we cannot directly use the procedure which was developed in Sec. II, where we assumed that the advecting velocity contains only the contribution of scales larger than L. Nevertheless, the nonlinear problem can be solved almost by the same method as in Sec. II. Physically it seems quite natural, since we deal with the advection of the vorticity $\omega(\mathbf{r})$ by the velocity of the eddies of scales larger than r. Formally it is related to the logarithmic character of the solution to be obtained.

First, we introduce the function $\boldsymbol{\varrho}$ which describes the distance between two fluid particles, one of which is placed at the origin of the qL reference frame:

$$\partial_t \boldsymbol{\varrho} = \mathbf{v}(t, \boldsymbol{\varrho}) - \mathbf{v}(t, 0) . \tag{3.2}$$

Further, we will be interested in the function $\boldsymbol{\varrho}(t_1, t, \mathbf{r})$ determined by (3.2) and by the initial condition $\boldsymbol{\varrho}(t_1, t_1, \mathbf{r}) = \mathbf{r}$ [more precisely it is the terminal condition since we will consider the solution of (3.2) at $t < t_1$]. Now for the function $\tilde{\omega}(t, \mathbf{r}) = \omega(t, \boldsymbol{\varrho}(t_1, t, \mathbf{r}))$, we find from (1.3) that $\partial_t \tilde{\omega} = \phi(t, \boldsymbol{\varrho})$. Solving this equation and using the initial condition $\omega(t_1, \mathbf{r}) = \tilde{\omega}(t_1, \mathbf{r})$, we obtain

$$\omega(t_1, \mathbf{r}) = \int_{-\infty}^{t_1} dt\, \phi(t, \boldsymbol{\varrho}(t_1, t, \mathbf{r})) . \tag{3.3}$$

This expression is a generalization of (2.3) for the nonlinear problem. Let us stress that the relation (3.3) is formally exact. It expresses the fact that the vorticity is a tracer in an inviscid flow.

To extract some additional information from the equation (3.2), we will make use of a slow (logarithmic) dependence of vorticity correlation functions on $\mathbf{r}$. The velocity difference in (3.2) can be written as the integral $v_\alpha - v_{0\alpha} = \int dr'_\beta \nabla_\beta v_\alpha$, where the integral is taken along a curve connecting the point 0 to the point $\boldsymbol{\varrho}$. With the logarithmic accuracy, the $\mathbf{r}'$-dependent quantity $\nabla_\beta v_\alpha$ in the above integral can be substituted by its value at $\mathbf{r}' = \boldsymbol{\varrho}$. Therefore, we arrive at the equation

$$\partial_t \varrho_\alpha(t_1, t) = \sigma_{\alpha\beta}(t, \boldsymbol{\varrho}) \varrho_\beta , \tag{3.4}$$

analogous to (2.4). The only difference is that now $\sigma_{\alpha\beta} = \nabla_\beta v_\alpha$ is $\boldsymbol{\varrho}$ dependent, though this dependence is logarithmic, that is very weak. Therefore, as well as in the passive problem, the Eq. (3.4) leads to the exponential growth of $\boldsymbol{\varrho}$ at decreasing t (what corresponds to increasing $t_1 - t$), but the expression for the $\ln(\varrho/r)$ will have the time dependence different from the passive case.

Note that turbulent motions of different scales contribute to the dispersion of the particles. Scales smaller than ϱ lead to a turbulent diffusion (with separation distance growing as a square root of time) while random advection by larger scales leads to an exponential in time growth of the distance. For the solution to be obtained, the main contribution into Lagrangian dispersion will be due to advection by large scales.

Let us turn to the correlation functions of ω. The direct generalization of the procedure developed in Sec. II is impossible. The problem is that at deriving the expressions of the (2.6, 2.14) type, we performed the independent averaging over the statistics' of the pumping "force" ϕ and of the velocity derivatives $\hat{\sigma}$. It is obviously incorrect in the nonlinear case since $\hat{\sigma}$ in (3.4) is induced by the vorticity itself and, therefore, its statistics is connected with the statistics of the force ϕ pumping the vorticity. The expressions (2.6, 2.14) are linear with respect to P_2, while we expect the vorticity correlation function to depend on the pumping rate in a nonlinear way. To get similar expressions, we consider the *variations* of the correlation functions with respect to the enstrophy production rate P_2 (and other production rates P_n). The formulas for the variations, analogous to (2.6, 2.14, 2.17), are derived in Appendix A as some averages over the statistics of $\hat{\sigma}$. Of course, now these are not explicit expressions for the vorticity correlation functions but complicated integral relations. Nevertheless, those relations enable us to establish the character of $\mathbf{r}$ dependence of the correlation functions.

We are starting with the variation of the pair correlation function F of ω which in accordance with the results of Appendix A is

$$\delta F(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = \delta P_2 \int dt\, \langle \chi(\rho(t)) \rangle_\sigma , \tag{3.5}$$

where $\boldsymbol{\rho} = \boldsymbol{\varrho}(t_1, t, \mathbf{r}_1) - \boldsymbol{\varrho}(t_2, t, \mathbf{r}_2)$, and the integration over t in (3.5) is performed from $-\infty$ to $\min(t_1, t_2)$. The function χ figuring in (3.5) is defined as in (2.5) since the pumping ϕ could be assumed δ correlated in the locally comoving reference frame. One should find the integral over t in r.h.s. of (3.5) at a given realization of the field $\hat{\sigma}$ and then average over the statistics of $\hat{\sigma}$. As it was in Sec. II where the passive scalar problem was treated, the integral $\int dt \chi$ is equal to the time $\tau_\star$ which is necessary for ρ to grow up to L at increasing $t_1 - t$:

$$|\boldsymbol{\rho}(\min(t_1, t_2) - \tau_\star)| = L . \tag{3.6}$$

Then we obtain from (3.5),

$$\delta F = \delta P_2 \langle \tau_\star \rangle_\sigma . \tag{3.7}$$

Therefore, before examining correlation functions of ω, we should establish the dynamic equation for $\boldsymbol{\rho}$. The evolution of the vector $\boldsymbol{\rho}$ figuring on the r.h.s. of (3.5) is determined by (3.4). If the difference $\boldsymbol{\varrho}_1 - \boldsymbol{\varrho}_2$ is not small in comparison with $\max(\varrho_1, \varrho_2)$, we find from (3.4),

$$\partial_t \rho_\alpha = \sigma_{\alpha\beta}(t, \boldsymbol{\rho}) \rho_\beta . \tag{3.8}$$

Indeed if $\boldsymbol{\varrho}_1 \sim \boldsymbol{\varrho}_2$ the expressions for $\hat{\sigma}(t, \boldsymbol{\varrho}_1)$ and $\hat{\sigma}(t, \boldsymbol{\varrho}_1)$ practically coincide due to logarithmic character of $\hat{\sigma}$ and can be substituted by $\hat{\sigma}(t, \boldsymbol{\rho})$; if $\boldsymbol{\varrho}_1 \ll \boldsymbol{\varrho}_2$ or $\boldsymbol{\varrho}_1 \gg \boldsymbol{\varrho}_2$ the equation for $\boldsymbol{\rho}$ will coincide with the Eq. (3.4) for the larger value and we again return to (3.8). A solution of the Eq. (3.8) should be obtained in two steps. Let $t_1 > t_2$, then we should first find the solution of the Eq. (3.4) in the interval $t_1 > t > t_2$ with the initial condition $\boldsymbol{\varrho}_1(t_1, t_1) = \mathbf{r}_1$ and then we should solve the Eq. (3.8) for $t < t_2$ with the initial condition $\boldsymbol{\rho}(t_2) = \boldsymbol{\varrho}_1(t_1, t_2) - \mathbf{r}_2$. Then we can find $\tau_\star$ from the relation (3.6).

To find $\tau_\star$, explicitly, we shall assume that the velocity gradients are long correlated and then demonstrate it self-consistently on a solution found below. That assumption is quite natural after it has been demonstrated in Sec. II that the correlation time τ of the passive scalar θ is logarithmically large in the comoving reference frame. The same is true also for the vorticity ω (as it will be clear from the solution to be obtained), although the dependence of the correlation time τ on the logarithm does not coincide with the dependence of the correlation time of the passive scalar θ. Since ω is advected by the velocity (3.1) induced by the vorticity itself it is natural to expect that the velocity gradients $\sigma_{\alpha\beta}$ have the same correlation time as ω. This cannot be taken for granted since ω is a Lagrangian invariant so it's correlation time in the comoving reference frame might be anomalously large compared to the correlation time of other velocity derivatives. We prove that for a logarithmic regime it is not the case. Starting from (3.1), we show that the correlation time of the velocity gradients $\sigma_{\alpha\beta}$ indeed coincides with the correlation time τ of ω and is, consequently, logarithmically large. This proof can be found in Appendix B. Here is an important difference from the passive regime considered in Sec. II where the correlation times of θ and σ are unrelated. Note that the assumption that the strain is slowly varying along a particle path with respect to the vorticity gradients has been formulated earlier by Weiss [18]. We shall see below that the assumption is statistically correct.

We thus deal with the case of a long correlated $\hat{\sigma}$ that has been analyzed in Sec. II for the passive scalar problem. It is convenient as in Sec. II to introduce three scalar functions: $\sigma_{xx} = -\sigma_{yy} = a$, $\sigma_{xy} = b + c$, and $\sigma_{yx} = b - c$, which are now $\mathbf{r}$ dependent. Solving the Eq. (3.8) for the simultaneous case $t_1 = t_2$, we find as for the passive problem

$$\tau_\star = \int_0^{\ln(L/r)} d\xi/\sqrt{a^2+b^2-c^2} + t_< \,, \tag{3.9}$$

where $r = |\mathbf{r}_1 - \mathbf{r}_2|$, $\xi = \ln(L/\rho)$ (recall that now a, b, c are ξ dependent) and $t_<$ is the total time when $a^2 + b^2 < c^2$. Consider now the different-time correlation function regarding $r_1 \sim r_2 \sim r$ and $t_1 > t_2$. If the difference $t_1 - t_2$ is larger than the characteristic time $\sim a^{-1}, b^{-1}$ than the value of $\boldsymbol{\rho}$ coincide with $\boldsymbol{\varrho}_1$ since at any given time t, $\varrho_1 \gg \varrho_2$. Solving (3.4), we can determine $\varrho(t)$ and then $\tau_\star$ from (3.6). We can do this by finding the time needed for ϱ_1 to change from r to L and subtracting $t_1 - t_2$ since for the case $t_1 > t_2$ the time $\min(t_1, t_2)$ in (3.6) is t_2. The result differs from that given by (3.9) in the value $t_1 - t_2$. Using now the relation (3.5), we find

$$\delta F(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2)$$
$$= \delta P_2 \left\langle \int_0^{\ln(L/r)} \frac{d\xi}{\sqrt{a^2+b^2-c^2}} + t_< - |t_1 - t_2| \right\rangle_\sigma . \tag{3.10}$$

Let us now turn to higher-order correlation functions. In this section, we will treat only even correlation functions of the vorticity ω since odd correlation functions are small in comparison with even ones, as it will be demonstrated in Sec. IV. This means that the probability distribution function $\mathcal{P}(\omega)$ can be regarded as an even functional of ω in the principal logarithmic approximation.

The direct generalization of the above scheme enables us to express the variations over P_2 of the high-order correlation functions. Consider as an example the fourth-order correlation function,

$$F_4 = \langle \omega(t_1, \mathbf{r}_1)\omega(t_2, \mathbf{r}_2)\omega(t_3, \mathbf{r}_3)\omega(t_4, \mathbf{r}_4)\rangle \,.$$

As it is demonstrated in Appendix A the variation of F_4 over P_2 is

$$\delta F_4(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2, t_3, \mathbf{r}_3, t_4, \mathbf{r}_4)$$
$$= \delta P_2 \left\langle \omega_3 \omega_4 \left(\int_0^{\ln(L/r)} \frac{d\xi}{\sqrt{a^2+b^2-c^2}} + t_< - |t_1 - t_2| \right) \right\rangle_\sigma + \cdots , \tag{3.11}$$

where r and $t_<$ are related to the points $\mathbf{r}_1$ and $\mathbf{r}_2$, they are defined as in (3.10), the dots in (3.11) designate the sum of terms which can be obtained from the one presented in r.h.s. of (3.11) by making permutations of the subscripts $1, 2, 3, 4$. The relations analogous to (3.11) can also be formulated for higher-order correlation functions: they will differ from (3.11) only by the number of ω fields appearing as the factors at the integral and by the number of permutations.

The r.h.s. of those relations are averages over the statistics of $\hat{\sigma}$ which is reduced to the statistics of ω and, consequently, they could, in principle, be expressed via the whole set of correlation functions of ω. Therefore, we have arrived at an infinite set of the integrodifferential equations which cover the whole set of correlation functions. The differential (with respect to P_2) equations should be solved with zero initial conditions at $P_2 = 0$ assuming fluid to be at rest without pumping. Note that the solution will depend only on P_2 but not on high-order pumping rates P_n. We will justify this in Sec. IV where we will demonstrate that contributions to the correlation functions related to P_n are small in comparison with terms depending only on P_2.

Thus, we formulated a possible way to find the probability distribution function $\mathcal{P}(\omega)$. Of course, it is impossible to reconstruct $\mathcal{P}(\omega)$ explicitly but we can establish some of its general homogeneity properties. We see that the relations (3.10, 3.11) as well as all the corresponding relations for high-order correlation functions are invariant under rescaling

$$\xi \to Z\xi; \quad \omega \to Z^{1/3}\omega; \quad t \to Z^{2/3}t \,, \tag{3.12}$$

what implies $a, b, c \to Z^{1/3}a, Z^{1/3}b, Z^{1/3}c$. For a self-similar probability distribution, we can establish the dependences of all correlation functions on ξ, e.g., $\langle aa\rangle \propto \xi^{2/3}$. Moreover, since only one dimensional constant P_2 enters our solution, we can estimate any even correlation function up to a dimensionless factor. These estimates can be extracted from the relation

$$\omega \sim (P_2\xi)^{1/3} \,, \tag{3.13}$$

determining the characteristic value of the vorticity on the scale $r = L\exp(-\xi)$. For example, the pair correlation function $\langle \omega_1\omega_2\rangle \sim (P_2\xi)^{2/3}$. That has a simple physical meaning: vorticity spectral transfer is due to a renormalized strain so that, similar to (2.14): $F = \langle\omega_1\omega_2\rangle = P_2\xi/\bar{\lambda}(\xi)$. Here $\bar{\lambda}(\xi)$ is the effective stretching rate which can be estimated as $\bar{\lambda}(\xi) \sim (P_2\xi)^{1/3}$ according to (2.13).

The estimate for the correlation time τ can be obtained from (3.10): $\tau \sim (P_2\xi)^{2/3}$, which means $\bar{\lambda}\tau = \xi$ as well as for the passive scalar problem. The above law of the stretching rate renormalization leads to the following behavior of the distance between fluid particles: $\ln(\varrho/r) \sim P_2^{1/2}t^{3/2}$. The time of the spectral transfer τ_* is then $\tau_* = \langle\tau_*\rangle_\sigma \sim P_2^{-1/3}\xi^{2/3}$, where $\xi = \ln(L/r)$. We see that the time of the spectral transfer τ_* is of the order of the correlation time τ. This means, in particular, that the velocity gradients (strain and vorticity) produced by the motions of different scales are strongly correlated in the locally comoving reference frame. That (together with the analysis in Appendix B) gives a long-expected theoretical substantiation for the experimental observation of Townsend [15] that the time scale of change of the rates of strain and of the directions, relative to the fluid, of the principal strain axes is large compared to the time scale of straining. The persistence of the strain is also suggested by numerous pictures [from the early ones [19] to the recent ones [20], Fig. 1(b)] that show that straining produces long thin streaks which do not show the small-scale wriggles. Physically, that seems to be connected with the fact that isovorticity lines tend to set themselves locally along the direction of the positive rate of strain as well as material lines in the passive scalar problem as it was described by Batchelor [11]. It was a 3D case that was studied in [15], while two-dimensional flows were considered in [19–21]. The reasons for the persistence of strain should be qualitatively the same in two and three dimensions, yet in our case the correlation time grows with the Reynolds number logarithmically but not by a power law as was supposed in [11]. Certainly, the strain is not slow everywhere in space [see, e.g., [21], Fig. 3(c)], what we prove is that the strain is long correlated in the regions that give the main contribution into the vorticity cascade. Our proof is statistical. The dynamics responsible for that is probably related to the fact that regions around stagnation points should give the main contribution into straining. The curvature of the pressure field is nearly isotropic in those regions which guarantees that velocity gradients vary slowly with respect to the vorticity gradients [21].

The coincidence of τ and τ_* implies also that there is no reason for the statistics of ω to be Gaussian. Indeed, the local Gaussianity of a passive scalar θ was found in Sec. II to appear at $\tau_* \gg \tau_0$, where τ_0 was the correlation time of $\hat{\sigma}$. The statistics of ω appears, thus, to be essentially non-Gaussian due to substantial fluctuations of the stretching rate. Nevertheless, using the above arguments leading to (3.13) it is possible to estimate up to a dimensionless factor

$$F_{2n} = \langle\omega_1\omega_2\cdots\omega_{2n}\rangle \sim F^n \sim (P_2\xi)^{2n/3}, \qquad (3.14)$$

for even correlation functions F_{2n} of the vorticity. This estimate resembles the Gaussianity property proven for the passive problem (where it was possible to find the numerical factors too). The relation (3.14) means that the reducible and irreducible contributions to F_{2n} are of the same order.

Let us now discuss the role of isolated vortices. They enter the formalism via the value $t_<$ in (3.9–3.11). On the solution (3.12–3.13), the renormalized mean strain and vorticity grow by the same law and so their correlation times; if there exist small-scale isolated vortices, their measure should not grow with ξ since $t_<$ has the same scaling $\xi^{2/3}$ as τ_*. If we assume for a moment that $t_< \sim \xi^a$ with $a > 2/3$, we come to a contradiction. Indeed, it follows from (3.9) that $\langle\omega\omega\rangle \sim \xi^a$ and the strain correlation function is also $\sim \xi^a$ according to Appendix B. Therefore, $\bar{\lambda} \sim \xi^{a/2}$ then the total time of passing from ξ to 1 is $\xi/\bar{\lambda} \sim \xi^{1-a/2}$ which could not be less than it's part $t_<$. The formalism developed allows us also to see that ξ dependencies of the correlation functions are insensible to the possible presence of coherent large-scale vortices. The (yet unknown) statistics of such vortices could influence only the ξ-independent part of $t_<$. At sufficiently small scales, the first ($\sim \xi^{2/3}$) term in the r.h.s. of (3.9–3.11) will be the main one and will determine the correlation functions.

We have considered a solution dependent only on the enstrophy production rate P_2 neglecting the existence of high-order pumping rates P_n. The justification of this approach as well as the behavior of odd correlation functions of ω are discussed in the next section.

IV. HIGH-ORDER INTEGRALS OF MOTION AND ODD CORRELATION FUNCTIONS

Let us emphasize that the above statement on non-Gaussianity has nothing to do with high-order integrals of motion. One might think that because of the conservation of arbitrary power of vorticity, the $2n$th correlation function are determined by the input of the $2n$th integral of motion and are independent of lower moments. This is not the case since only the flux of squared vorticity is constant in the inertial interval while higher fluxes grow with k due to a contribution to their pumping from lower moments (the phenomenon of "distributed pumping" [10,22]). Here, we describe that phenomenon.

For the flux of ω^2 stuff in the inertial interval, one gets in a Kolmogorov manner

$$\langle[(\mathbf{v}_1\cdot\nabla_1) + (\mathbf{v}_2\cdot\nabla_2)]\omega_1\omega_2\rangle = \langle\phi_1\omega_2 + \phi_2\omega_1\rangle = P_2 \,. \qquad (4.1)$$

The rhs of (4.1) is constant at $|\mathbf{r}_1 - \mathbf{r}_2| \ll L$, this constant is equal simply to the pumping rate of ω^2. For ω^4 one gets similarly,

$$\langle[(\mathbf{v}_1\cdot\nabla_1) + (\mathbf{v}_2\cdot\nabla_2)]\omega_1^2\omega_2^2\rangle = \langle\phi_1\omega_1\omega_2^2 + \phi_2\omega_2\omega_1^2\rangle \,. \qquad (4.2)$$

Besides the irreducible part which is the constant P_4 in the inertial interval, the correlator in the rhs of (4.2) necessarily contains the reducible part,

$$2\langle\phi_1\omega_2\rangle\langle\omega_1\omega_2\rangle+2\langle\phi_2\omega_1\rangle\langle\omega_1\omega_2\rangle=2P_2\langle\omega_1\omega_2\rangle,$$

that changes with $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$ as the pair correlator F. Since our pair correlation function F grows as r_{12} decreases, then for sufficiently small r_{12} one can neglect the constant irreducible contribution determined by P_4 in comparison with $2P_2F$. The analogous analysis can be performed for all the even fluxes, these fluxes being expressed in terms of P_2:

$$\langle[(\mathbf{v}_1\cdot\nabla_1)+(\mathbf{v}_2\cdot\nabla_2)]\omega_1^n\omega_2^n\rangle\sim P_2^n\ln^{2(n-1)/3}(L/r_{12})\,. \tag{4.3}$$

Note that for $n>2$, the coefficient in (4.3) in contrast to the case of the fourth-order correlation function cannot be established in the explicit form.

Thus, the fluxes of the high-order integrals of motion are nonconstant in the inertial interval. It is worth emphasizing that the reason for this phenomenon is the presence of the effective forcing for higher integrals "distributed" over scales of the direct cascade which means the absence of the inertial intervals for the integrals. Let us stress that the flux change has nothing to do with nonconservation of the integrals. The enstrophy production rate P_2 thus determines the whole set of the fluxes. This is actually a justification of the treatment of Sec. III where we neglected all high-order integrals. The possibility to do so is also confirmed on the diagrammatic language in Appendix A.

Let us turn now to the investigation of the odd correlation functions of the vorticity. First, we should consider the contributions of the same type as for the even correlation functions, these contributions being related to the odd correlation functions of the pumping force ϕ. We introduce the third-order correlation function,

$$\langle\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\phi(t_3,\mathbf{r}_3)\rangle$$
$$=P_3\delta(t_1-t_2)\delta(t_1-t_3)\chi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\,, \tag{4.4}$$

where $\chi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ is actually a function of differences $\mathbf{r}_i-\mathbf{r}_j$, which tends to zero if any $|\mathbf{r}_i-\mathbf{r}_j|$ tends to ∞. The characteristic scale where this diminishing begins is the correlation length L of the pumping force ϕ. We assume also that $\chi(0,0,0)=1$. Then P_3 in (4.4) is the pumping rate of ω^3. With our logarithmic accuracy we can take $\chi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\theta(L-r_{12})\theta(L-r_{13})\theta(L-r_{23})$, where θ is the step function. By the methods of Appendix A, it is possible to derive for the variation of the third-order correlation function $F_3=\langle\omega_1\omega_2\omega_3\rangle$ over P_3 the formula of (3.10) type

$$\delta F_3=\delta P_3\int dt\,\langle\chi_3(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,\boldsymbol{\varrho}_3)\rangle_\sigma\,, \tag{4.5}$$

where $\boldsymbol{\varrho}_i=\boldsymbol{\varrho}(t_i,t,\mathbf{r}_i)$ and the integration in (4.5) is performed from $-\infty$ up to $\min(t_1,t_2,t_3)$. The integral over time in (4.5) can be again estimated as $\xi/\bar\lambda$ where $\bar\lambda\sim(P_2\xi)^{1/3}$ is the effective stretching rate. Then we get after integration of (4.5) over dP_3,

$$F_3\sim P_3\xi/\bar\lambda\sim(P_3/P_2^{1/3})\xi^{2/3}\,. \tag{4.6}$$

We see that for large $\xi=\ln(L/r)$ this contribution is small in comparison with the "normal" value $P_2\xi$ following from (3.13).

The third-order term (4.4) give rise also to the contributions to all other odd correlation functions. The contribution to the $2n+1$-th order correlation function of ω is estimated as $F_{2n+1}\sim F_3F_2^{n-1}$, which follows from the generalization of (4.5) leading to the relations of (3.11) type. We see that this contribution is again smaller in $\xi^{-1/3}$ than the "normal" estimation $(P_2\xi)^{2n+1}$ that would follow from (3.13). If we estimate the contributions to correlation functions F_{2n+1} associated with the high-order pumping rate P_m, then we conclude that the contributions contain additional powers of the small parameter $\xi^{-1/3}$ in front of $(P_2\xi)^{2n+1}$ and can consequently be neglected (see also Appendix A). We conclude that the contributions to the odd correlation functions of the vorticity ω related to high-order production rates P_m are small in comparison with the even correlation functions, this smallness can be estimated as $\xi^{-1/3}$.

Because of this smallness the question arises: does P_3 actually determine the main contribution into F_{2n+1}? Already from (4.1) it is clear that there are contributions into the third-order correlation function that are nonzero even if the third-order correlation function of the pumping is zero. To evaluate such contribution independent on P_3, we rewrite (4.1) in the following form:

$$\nabla_{1\alpha}\int_2^1 dr_\beta\langle\sigma_{\alpha\beta}\omega_1\omega_2\rangle=P_2\,, \tag{4.7}$$

where integration is performed along any curve going from $\mathbf{r}_2$ to $\mathbf{r}_1$, and $\sigma_{\alpha\beta}$ depends on the point $\mathbf{r}$ on this curve. The solution of this equation is written as

$$\int_2^1 dr_\beta\langle\sigma_{\alpha\beta}\omega_1\omega_2\rangle=(1/2)P_2\rho_\alpha+\epsilon_{\alpha\beta}\nabla_\beta\Phi(\boldsymbol{\rho})\,, \tag{4.8}$$

where $\boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2$. The second term in the r.h.s. of (4.8) corresponds to the above contribution (4.6). Indeed, this contribution is related to large scales and is consequently logarithmically dependent on scales. This means that the contribution (4.6) gives $\langle\sigma_{\alpha\beta}\omega_1\omega_2\rangle\simeq(1/2)F_3\epsilon_{\alpha\beta}$ since this average should be isotropic and the term proportional to $\delta_{\alpha\beta}$ is absent since $\sigma_{\alpha\beta}$ is traceless. After integration we find,

$$\int_2^1 dr_\beta\langle\sigma_{\alpha\beta}\omega_1\omega_2\rangle\to(1/4)\epsilon_{\alpha\beta}\nabla_\beta(F_3\rho^2),$$

which corresponds to the last term in r.h.s. of (4.8). The first term in the r.h.s. of (4.8) means that there exists the ρ-independent contribution to F_3, this is just the contribution which we are looking for since it is determined solely by P_2. If P_3 is absent then $F_3\simeq P_2$, i.e., even smaller in logarithmic parameter than the contribution related to P_3.

Analogously the higher-order odd correlation functions of ω can be examined. For this we should start from the expression (4.3) and produce the same analysis as for F_3. We thus get $F_{2n+1} \propto \xi^{2(n-1)/3}$ if the odd correlation functions of the pumping are absent and $F_{2n+1} \propto \xi^{2n/3}$ in a general case. We conclude that in any case, the odd correlation functions of ω are suppressed in comparison with the even ones. The above results enable us to find also the "distributed" fluxes of odd integrals of motion. Evaluating the average $\langle \phi_1 \omega_1^n \omega_2^n \rangle$ arising instead of r.h.s. of (4.2) for the $2n+1$-th flux, we find from (3.13, 4.1, 4.6) the estimate for this flux: $P_3(P_2\xi)^{2(n-1)/3}$. Of course for $n = 1$ this estimate gives the third-order flux P_3, for $n > 1$ this flux grows with decreasing scale. Therefore, the situation for odd fluxes is the same as for even fluxes: at $n > 1$ the "distributed" flux of the $2n+1$-th order is larger than the constant P_{2n+1} related to the direct pumping.

To conclude, the even correlation functions are determined by P_2, while the odd ones also by P_3.

CONCLUSION

We have found the correlation functions of the vorticity in 2D turbulence at the region of scales where the direct cascade exists. The true expressions for the correlation functions differ from those obtained from simple dimension estimates by logarithmic factors. In particular, the shape of the energy spectrum is as follows: $E(k) \sim P_2^{2/3} k^{-3} \xi^{-1/3}$ where $\xi = \ln(kL)$; it does not depend on a particular value of the unknown dimensionless constants. The same is true for higher-order correlation functions: we can find the ξ dependencies but not constant factors. The fact that the true form of the pair correlation function coincides with the result of the one-loop calculation means that there is no vertex renormalization for a logarithmic regime. Let us stress that our logarithmic solution does not depend of the statistics of the pumping. The probability distribution function depends only on the enstrophy production rate P_2 while the space and time dependencies of the correlation functions are universal, i.e., pumping independent.

Other hypothetical powerlike steady distributions [6,7,9] with the exponents larger than 3 correspond to the correlation functions of the velocity gradients that tend to constants as $\xi \to \infty$. This contradicts to the relation between τ_* and a, b and c given at Sec. III: if the correlation functions of $a, b,$ and c turned into constants as $\xi \to \infty$, then $\tau_* \propto \xi$ and the pair correlation function of the vorticity $F = P_2\xi$ would be logarithmic in contradiction with the initial assumption. Therefore, distributions with a nonlogarithmic vorticity correlation function cannot be turbulent steady solutions under the action of a general pumping. This has a clear physical meaning: if there existed some steeper spectrum $E(k) \propto k^{-x}$ with $x > 3$, then it would produce the main strain due to largest scales whatever the temporal correlations may be. Therefore, a small-scale vorticity behaves like a passive scalar in such a field and the appearance of the pumping with nonzero vorticity influx would produce a logarithmic regime which corresponds to $x = 3$. This follows from the statement that whatever the statistics of the large-scale velocity field may be, the mean stretching rate (the Lyapunov exponent) is positive so that the stretching is exponential and the correlation functions of the passive scalar are logarithmic [13]. Let us emphasize that the universality at small scales is proved only for steady turbulence under the action of pumping; it might be that decaying turbulence produces different small-scale asymptotics for different initial distributions [20].

Note that our logarithmic regime does not correspond to the absence of coherent vortices. We show that the stretching process is suppressed inside the vortices and that such "islands of vorticity" give negligible contribution into the correlation functions at small scales (see also [23]). The main contributions are produced by the "sea of strain" which renormalizes itself in a way described above. In addition to the present theory of the direct cascade, the future complete theory of two-dimensional turbulence should include also the description of the inverse cascade and coherent vortices.

The set of the correlation functions found corresponds to nonzero fluxes of all vorticity integral of motion though only the enstrophy flux is constant in the inertial interval while higher fluxes vary with the scale. Recent numerics [14] seem to confirm the prediction made first in [10]. Our picture is based on the assumption that P_2 is not small in comparison with another P_n. If there exists such n that $P_n \gg (P_2)^{n/2}$ then an intermediate asymptotic appears which requires a special consideration.

The account of viscosity could be readily incorporated in the above formalism. Viscosity makes the vorticity field to be smooth at $r \ll L_{vis}$ so that the pair correlation function, for instance, behaves as r^2 [4,11,12]. Yet, viscosity gives a negligible contribution into the correlation functions in the inertial interval. Formally, this is related to the fact that there are no ultraviolet divergences in the perturbation series for the vorticity correlation functions. If, however, one considers vorticity gradients, there should be an anomalous scaling related to the effects of viscosity and to ultraviolet divergences.

Our results allow also for the prediction on the correlation functions of a passive scalar θ advected by 2D turbulence with an arbitrary Prandtl number. Since the equation for θ formally coincides with the one for ω then the scale dependences of the correlation functions of θ and ω should coincide at the scales $L_{vis} \ll r \ll L$: $\langle \theta^n(\mathbf{r}_1)\theta^n(\mathbf{r}_2)\rangle \simeq [P_2 \ln(L/ \mid \mathbf{r}_1 - \mathbf{r}_2 \mid)]^{2n/3}$, etc. Here P_2 is the flux of the squared scalar.

ACKNOWLEDGMENTS

We thank R. Kraichnan, A. Migdal and A. Polyakov for the helpful discussions. Useful remarks of D. Khmelnitskii and A. Larkin are acknowledged. We are grateful for the support of the Rashi Foundation (G.F.) and of the Landau-Weizmann program (V.L.).

APPENDIX A: DIAGRAM REPRESENTATION OF CORRELATION FUNCTIONS

It is natural to investigate fluctuation effects in a system with many interacting degrees of freedom in terms of a diagrammatic technique. In this appendix, we will make use of the diagram technique especially adapted for hydrodynamical systems. Such a technique was first developed by Wyld [24], who studied velocity fluctuations in a 3D turbulent fluid. The next step was made by Martin, Siggia, and Rose [25], who generalized the Wyld technique for a broad class of dynamical systems. The textbook description of the diagram technique is presented in the monograph by Ma [26]. The diagram technique may be formulated in terms of path integrals as was first suggested by de Dominicis [27] and Janssen [28]. The textbook description of this method can be found in the monograph [29].

Let us remind the reader that we consider 2D turbulence in the region of the direct cascade. We will utilize the so-called quasi-Lagrangian variables [2,3] enabling one to avoid the masking effect of sweeping leading to infrared divergences in the original Wyld diagram technique. The diagram technique in qL variables is generated by the effective action $I = I_{(2)} + I_{int}$, where

$$I_{(2)} = \int dt\, d^2r\, p\, \partial_t\, \omega + \frac{i}{2} P_2 \int dt\, d^2r_1\, d^2r_2\, p_1 p_2 \chi(\mathbf{r}_1 - \mathbf{r}_2)\,, \qquad \text{(A1)}$$

$$I_{int} = \int dt\, d^2r\, p(\mathbf{v} - \mathbf{v}_0)\nabla\omega\,, \qquad \text{(A2)}$$

constructed in accordance with (1.3). In (A2) $\mathbf{v}$ is the qL-velocity, $\partial_t \equiv \partial/\partial t$, the subscript 0 denotes the velocity taken at the origin, $\omega = \epsilon_{\alpha\beta}\nabla_\alpha v_\beta$ is the vorticity and p is an auxiliary field enabling us to express susceptibilities to the external force acting on the system in terms of correlation functions. For example, the Green's function G is the pair correlation function: $G = -\langle \omega p\rangle$. The function $\chi(\mathbf{r}_1 - \mathbf{r}_2)$ in (A1) represents the pair correlation function of the pumping "force" ϕ figuring in (1.3) which is explained in the Introduction to be δ correlated in time (in qL variables). The function χ is normalized by the condition $\chi(0) = 1$, then P_2 in (A1) is the production rate of enstrophy. Now all correlation functions can be written in the form of functional integrals, for instance,

$$F = \langle \omega_1 \omega_2\rangle \equiv \int \mathcal{D}\omega \mathcal{D}p\, \exp(iI)\omega_1\omega_2\,, \qquad \text{(A3)}$$

where the integration $\mathcal{D}\omega\mathcal{D}p$ is performed over all functions $\omega(t,\mathbf{r})$, $p(t,\mathbf{r})$ [25]. Note that $\langle pp\rangle \equiv 0$, the analogous property is true for higher-order correlation functions of the field p.

The structure of (A2) shows that we should treat correlation functions of the difference $v_\alpha - v_{0\alpha}$. Consider as an example the pair correlation function $\langle (v_{1\alpha} - v_{0\alpha})(v_{2\beta} - v_{0\beta})\rangle$, where the subscripts 1 and 2 mean that the corresponding velocities should be taken at the points $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. In the main logarithmic approximation the correlation function can be written as

$$\langle (v_{1\alpha} - v_{0\alpha})(v_{2\beta} - v_{0\beta})\rangle = \frac{1}{4}(\delta_{\alpha\beta}\mathbf{r}_1\cdot\mathbf{r}_2 - r_{1\beta}r_{2\alpha})F + \frac{1}{4}(\delta_{\alpha\beta}\mathbf{r}_1\cdot\mathbf{r}_2 - r_{1\alpha}r_{2\beta} + r_{1\beta}r_{2\alpha})F_s\,, \qquad \text{(A4)}$$

where F is defined by (A3) and F_s designates the pair correlation functions of the strain $s_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha$:

$$F_s(\mathbf{r}_1, \mathbf{r}_2) = \langle s_{xx}(\mathbf{r}_1)s_{xx}(\mathbf{r}_2)\rangle \quad , \qquad \text{(A5)}$$

where s_{xx} designates the component of the strain along the axis X. Utilizing the expression (A4), we can find that in the main logarithmic approximation,

$$\langle (v_{1\alpha} - v_{0\alpha})\omega_2\rangle = -(1/2)\epsilon_{\alpha\beta}r_{1\beta}F\,. \qquad \text{(A6)}$$

Analogously the expression for the following pair correlation function can be obtained:

$$\langle (v_{1\alpha} - v_{0\alpha})p_2\rangle = (1/2)\epsilon_{\alpha\beta}r_{1\beta}G\,. \qquad \text{(A7)}$$

The above relations mean that in the principal logarithmic approximation the difference $\mathbf{v} - \mathbf{v}_0$ can be written as

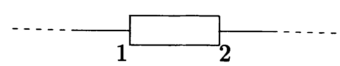
$$v_\alpha - v_{0\alpha} \to -(1/2)\epsilon_{\alpha\beta}r_\beta\omega + s_{\alpha\beta}r_\beta = \sigma_{\alpha\beta}r_\beta\,, \qquad \text{(A8)}$$

where $\sigma_{\alpha\beta} = \nabla_\beta v_\alpha(\mathbf{r})$. The relations between the correlation functions of $s_{\alpha\beta}$ and ω are treated in Appendix B.

Starting from (A1, A2) and definitions of the type of (A3), we can formulate the conventional perturbation series for the correlation functions of ω and p, e.g., for F or G. The series is produced by the expansion of $\exp(iI)$ in I_{int}, given by (A2). It will be also convenient to expand $\exp(iI)$ in $P_2\chi$. Then the objects which arise in the perturbation series are the products $P_2\chi$ and the bare Green's functions, the latter function being

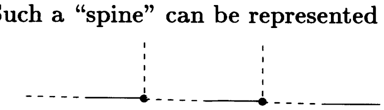
$$G_0(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = -i\theta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2)\,, \qquad \text{(A9)}$$

where $\theta(x)$ is the step function. The terms of the series can be represented by Feynman diagrams, where lines will correspond to the Green's functions (A9) only (since $F_0 = 0$ in this case) and the vortices are determined by (A2). We will designate the Green's function by the combined dashed-solid line where the solid part corresponds to the field p and the dashed part corresponds to the field ω or $s_{\alpha\beta}$. The Green's function can be attached to the product $P_2\chi$, which will be designated by the rectangular. Then the first contribution to the pair correlation function F can be represented as



where the integration is implied over $t_1 = t_2, \mathbf{r}_1, \mathbf{r}_2$. The high-order diagrams contain the interaction vertex determined by (A2).

Consider the diagrammatic series for the dressed (whole) Green's function G, which is a function of the time difference $t_1 - t_2$ and of both coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ since we lost space homogeneity at passing to qL variables. A diagram for G can be presented as the "spine" constructed from G_0 lines only and the "ribs" made from G_0 lines started from $P_2\chi$ rectangles and finished at the "spine." Such a "spine" can be represented as



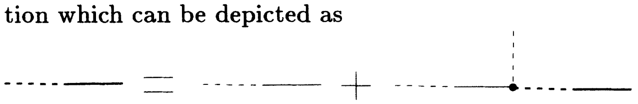
where only the piece of a diagram adjoined to the "spine" is depicted.

We see that the calculation of G can be performed in two steps: first we should perform the summation of diagrams with different "ribs" attached to a given "spine" and then sum over all "spines." The first step (summation of "ribs") is reduced to averaging of a given "spine" with attached ω or $s_{\alpha\beta}$ fields over the statistics of these fields. We can inverse the order of these operations and find first $\mathcal{G}$ which is the sum of all "spines" with the attached fields and then average the result over the statistics of these fields. It can be written as

$$G(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = \langle \mathcal{G}(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) \rangle_\sigma . \tag{A10}$$

The object $\mathcal{G}$ can be interpreted as the Green's function of a passive scalar in a given external velocity field characterizing by the velocity gradients $\hat{\sigma}$.

The calculation of $\mathcal{G}$ is reduced to the summation of the "ladder" sequence of diagrams of the type depicted above. This summation leads to the diagrammatic equation which can be depicted as



where the thick combined line designates the function $\mathcal{G}$, the thin combined line designates the bare function (A9) and the vertical dashed line designates the "external" field $\sigma_{\alpha\beta}$ [over which the averaging in (A11) should be performed]. This equation can be written analytically, after the differentiation with respect to t_1 it takes the form

$$\frac{\partial}{\partial t_1} \mathcal{G}(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = -i\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2) - \left((v_{1\alpha} - v_{0\alpha})\nabla_{1\alpha} - \frac{1}{2}\nabla_{1\alpha}\omega\epsilon_{\alpha\beta}r_{1\beta} \right) \mathcal{G}(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) , \tag{A11}$$

where we have utilized the explicit expression (A9) and also (A8). To solve the equation (A11), note that the equation practically coincides with the Eq. (1.3), which has the formally exact solution (3.3). The only difference is in the term proportional to the vorticity gradient $\nabla_\alpha \omega$. Without this term, we find analogously to (3.3)

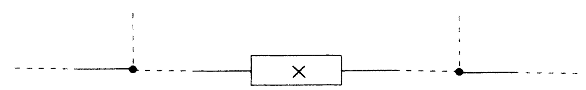
$$\mathcal{G}(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = -i\theta(t_1 - t_2)\delta(\boldsymbol{\varrho}(t_1, t_2, \mathbf{r}_1) - \mathbf{r}_2) . \tag{A12}$$

The term with $\nabla_\alpha \omega$ in (A11) can now easily be taken into account by introducing into r.h.s. of (A12) the extra factor $\mathcal{Y}$ controlled by the equation

$$\frac{\partial}{\partial t_1} \mathcal{Y}(t_1, t_2) = \frac{1}{2}\nabla_\alpha \omega(t_1, \boldsymbol{\varrho}_1)\epsilon_{\alpha\beta}\varrho_{1\beta}\mathcal{Y}(t_1, t_2) , \tag{A13}$$

with the initial condition $\mathcal{Y}(t, t) = 1$. A solution of the equation (A13) is an ordered exponent which contains $\nabla\omega$ in the argument. Because of the logarithmic character of the dependence of all the correlation functions of ω, the operator of the type $r\nabla$ figuring in (A13) kills one power of the logarithm in the correlation functions. Therefore, the effects related to $\mathcal{Y}$ can be neglected with our logarithmic accuracy and we will omit this factor below.

Now we aim at finding a representation for correlation functions of ω analogous to the expression (A10) for the Green's function. Unfortunately the direct generalization of this procedure is impossible: because of a nonlinear dependence of the correlation functions on the pumping, one cannot restrict the consideration to a single "spine." Nevertheless, we can arrive at the representation of the (A10) type but for variations of the correlation functions over P_2. Let us demonstrate this considering the diagrams, e.g., for δF. These diagrams will contain a marked rectangular corresponding to $\delta P_2 \chi$. We can find two "spines" started from this marked rectangular and going to the end points of a diagram. This situation is represented in the figure,



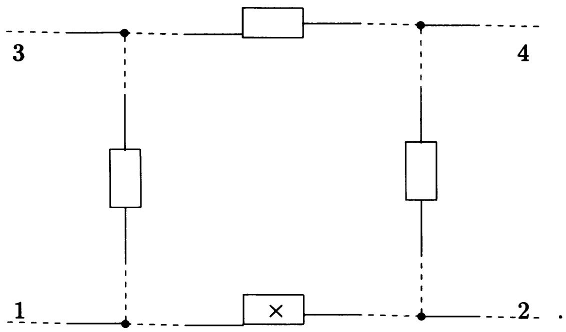
where the crossed out rectangle designates $\delta P_2\, \chi$. Now we see the reason for taking a variation: we obtain two marked "spines" attached to the marked rectangle.

As before we can calculate δF in two steps. The first step is the summation over "ribs" that reduces to the averaging over the statistics of the fields attached to the "spines," and the second step is the summation over all "spines." We again inverse the order, then the summation over "spines" will produce two $\mathcal{G}$ functions, attached to $\delta P_2 \chi$ and the result should be averaged over the statistics of $\sigma_{\alpha\beta}$. Using the explicit expression (A12), we find

$$\delta F(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = \delta P_2 \int dt \, \langle \chi(\rho) \rangle_\sigma , \tag{A14}$$

where $\boldsymbol{\rho} = \boldsymbol{\varrho}(t_1, t, \mathbf{r}_1) - \boldsymbol{\varrho}(t_2, t, \mathbf{r}_2)$ and the integration over t is performed from $-\infty$ to $\min(t_1, t_2)$. Of course for the case of P_2 independent $\hat{\sigma}$ the relation (A14) reproduces (2.6).

Let us make some remarks concerning higher-order correlation functions of ω. The variation of these correlation functions over P_2 can be found by direct generalization of the above method. Any diagram representing nth order correlation function $\langle \omega(t_1, \mathbf{r}_1)\omega(t_2, \mathbf{r}_2) \ldots \rangle$ has two "spines" attached to the marked rectangle and a number of "ribs," some "ribs" have "free" legs ending at the points $\mathbf{r}_k$. An example is given below where a diagram for a fourth-order correlation function F_4 is depicted



Here, the points 1 and 2 are the ends of the "spines" attached to the marked rectangle and the points 3 and 4 denote the ends of two "free" legs. As above, the summation of all the diagrams will give the result which can be thought of as being averaged over the statistics of $\hat{\sigma}$ of the sum of "spines." The sum of the spines will give the same term as in r.h.s. of (A14) and the presence of "free" legs leads to arising a product of ω taken in corresponding points. Thus, we come to the following expression for the variation of the fourth-order correlation function F_4 over P_2:

$$\delta F_4(t_1,\mathbf{r}_1,t_2,\mathbf{r}_2,t_3,\mathbf{r}_3,t_4,\mathbf{r}_4)$$
$$= \delta P_2 \left\langle \omega_3\omega_4 \int dt\, \chi(\boldsymbol{\rho}) \right\rangle_\sigma + \cdots , \quad \text{(A15)}$$

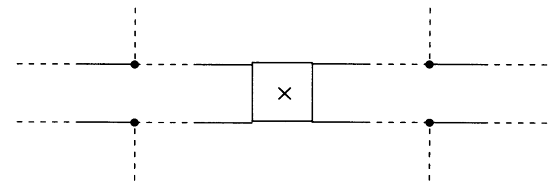
where $\boldsymbol{\varrho}$ is the same as in (A14) and ω_3 and ω_4 designate the values of ω taken in the points $\mathbf{r}_3$ and $\mathbf{r}_4$. The dots in (A15) designate the sum of terms which can be obtained from the one presented in r.h.s. of (A15) by the permutations of the subscripts $1,2,3,4$. The point is that there exist diagrams where "spines" end at two arbitrary points not only at 1 and 2 as in the presented figure. The relations analogous to (A15) can also be formulated for higher-order correlation functions: they will differ from (A15) only by the number of ω fields appearing as the factors at the integral and by the number of permutations.

Apart from pumping the enstrophy, the external force ϕ in (1.3) pumps also all higher-order integrals of motion P_n. Therefore, the question arises concerning respective contributions to the correlation functions. In our formalism, the presence of P_n means that, besides (A1, A2), we should take into account also the terms of higher order in p in the effective action. The term corresponding to P_4 has the following form:

$$I_{(4)} = \frac{i}{24} P_4 \int dt\, d^2r_1\, d^2r_2\, d^2r_3\, d^2r_4$$
$$\times p_1 p_2 p_3 p_4 \chi_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\,, \quad \text{(A16)}$$

where the function $\chi_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ is actually the function of the differences of coordinates, the characteristic scale of this function is L and $\chi_4(0,0,0,0)=1$. The contributions to the effective action associated with other integrals of motion P_n can be introduced in a similar way.

As above, it is possible to establish some relations for the variations of the correlation functions of ω over P_n. Let us consider the four-point correlation function $F_4 = \langle \omega_1\omega_2\omega_3\omega_4 \rangle$ as an example. Its variation over P_4 in the diagrammatic language can be represented schematically as follows:



Here the crossed out quadrangle designates the product $\delta P_4 \chi_4$, there are four "spines" attached to the quadrangle and a set of "ribs" attached to the "spines." As above, the summation over "ribs" is reduced to the averaging over the statistics of ω and, finally, we come to the expression

$$\delta F_4(t_1,\mathbf{r}_1,t_2,\mathbf{r}_2,t_3,\mathbf{r}_3,t_4,\mathbf{r}_4)$$
$$= \delta P_4 \int dt \langle \chi_4(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,\boldsymbol{\varrho}_3,\boldsymbol{\varrho}_4) \rangle_\sigma\,, \quad \text{(A17)}$$

where $\boldsymbol{\varrho}_1 = \boldsymbol{\varrho}(t_1,t,\mathbf{r}_1)$ and so further and the integration over the time t is performed from $-\infty$ to $\min(t_1,t_2,t_3,t_4)$. We see that the structure of (A17) is very close to one of (A14). The analogous expressions can be derived also for variations of other correlation functions of ω over all P_n. For example, the variation over P_3 is as follows:

$$\delta F_4(t_1,\mathbf{r}_1,t_2,\mathbf{r}_2,t_3,\mathbf{r}_3,t_4,\mathbf{r}_4)$$
$$= \delta P_3 \int dt \langle \omega_4 \chi_3(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,\boldsymbol{\varrho}_3) \rangle_\sigma + \cdots . \quad \text{(A18)}$$

The reason for arising $\omega_4 = \omega(t_4,\mathbf{r}_4)$ in (A18) is the same as in (A15), the dots in (A18) as in (A15) designate the sum of terms which can be obtained from the one presented in r.h.s. of (A18) by making permutations of the subscripts $1,2,3,4$.

Comparing (A17) with $\delta F_4/\delta P_2$, determined by (A15) one concludes using estimates (3.13) that the contribution to F_4 related to P_4 could be neglected at large enough logarithms. The same assertion is valid for all even correlation functions: the contributions to the correlation functions due to P_n are small at $n > 2$. This means that the terms $I_{(n)}$ in the effective action analogous to (A16) for $n > 2$ are small and we actually can restrict ourselves by the action (A1, A2). We already saw this by considering fluxes (4.3).

APPENDIX B: FROM VORTICITY TO STRAIN

In this appendix, we consider the transformation from the vorticity to strain correlation functions. The starting point of the transformation is the relation

$$v_\alpha(\mathbf{r}) = -\epsilon_{\alpha\gamma}\nabla_\gamma \int \frac{d^2R}{2\pi} \ln|\mathbf{r}+\mathbf{R}|\,\omega(\mathbf{R}), \tag{B1}$$

enabling one to reconstruct the velocity v_α from the vorticity ω. From (B1) it follows that the pair correlation function of the velocity derivatives $\sigma_{\alpha\beta}$ can be written in the following form

$$\langle \sigma_{\alpha\beta}(\mathbf{r}_1)\sigma_{\mu\nu}(\mathbf{r}_2)\rangle = \epsilon_{\alpha\gamma}\epsilon_{\mu\rho}\nabla_{1\gamma}\nabla_{1\beta}\nabla_{2\rho}\nabla_{2\nu}\Upsilon(\mathbf{r}_1,\mathbf{r}_2), \tag{B2}$$

where

$$\Upsilon(\mathbf{r}_1,\mathbf{r}_2) = \int \frac{d^2R_1}{2\pi}\int \frac{d^2R_2}{2\pi} \ln|\mathbf{R}_1-\mathbf{r}_1| \times \ln|\mathbf{R}_2-\mathbf{r}_2|\,F(\mathbf{R}_1,\mathbf{R}_2). \tag{B3}$$

Here $F(\mathbf{R}_1,\mathbf{R}_2) = \langle\omega(\mathbf{R}_1)\omega(\mathbf{R}_2)\rangle$ is the pair correlation function of the vorticity. In what follows, it is important that this function depends on coordinates via a logarithmic function. Our aim is to establish how the operations (B2–B3) influence that logarithmic dependence. We start from obtaining some general relations, then we consider the particular case of a simultaneous correlation function depending on $|\mathbf{R}_1-\mathbf{R}_2|$ and, finally, we consider most complicated case of a different-time correlation function. Our consideration culminates in the proof that the correlation times of the strain and vorticity are the same for logarithmic correlation functions.

The integration over angles in (B3) can be performed using the relations (see [30] 2.6.36.9 and 2.6.36.15)

$$\int_0^{2\pi} d\varphi \cos(m\varphi)\ln(R^2-2Rr\cos\varphi+r^2) = -\frac{2\pi}{m}\left(\frac{r}{R}\right)^m,$$

$$\int_0^{2\pi} d\varphi \ln(R^2-2Rr\cos\varphi+r^2) = 4\pi\ln R, \tag{B4}$$

where $m=1,2,\ldots$ and $R>r$. From these relations it follows for an arbitrary function f,

$$\int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \ln(R_1^2-2R_1r_1\cos\varphi_1+r_1^2) \times \ln(R_2^2-2R_2r_2\cos\varphi_2+r_2^2) f\left[\cos(\varphi_1+\varphi_2+\varphi)\right]$$
$$= 16\pi^2\ln R_1\ln R_2 f_0 + 8\pi^2\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{r_1r_2}{R_1R_2}\right)^m f_m\cos(m\varphi) \tag{B5}$$

where the inequalities $R_1>r_1$ and $R_2>r_2$ are implied and f_m are the Fourier components

$$f_m = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \exp(im\varphi) f(\cos\varphi).$$

Utilizing the relation (B5), we can rewrite the expression (B3) in the form of the series

$$\Upsilon(\mathbf{r}_1,\mathbf{r}_2) = \sum_{m=0}^{\infty}\Upsilon_m(\mathbf{r}_1,\mathbf{r}_2),$$

where

$$\Upsilon_0(\mathbf{r}_1,\mathbf{r}_2) = \int_0^\infty dR_1R_1\int_0^\infty dR_2R_2 \times \ln[\max(R_1,r_1)]\ln[\max(R_2,r_2)]\,F_0(R_1,R_2),$$

$$\Upsilon_m(\mathbf{r}_1,\mathbf{r}_2) = \int_0^\infty dR_1R_1\int_0^\infty dR_2R_2 \times\frac{1}{2m^2}(u_1u_2)^m F_m(R_1,R_2)\cos(m\varphi). \tag{B6}$$

Here φ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, $m=1,2,\ldots$, $u_1=r_1/R_1$ if $r_1<R_1$ and $u_1=R_1/r_1$ if $r_1>R_1$, and u_2 is analogously defined. The functions F_m are the Fourier transforms

$$F_m(R_1,R_2) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\exp(im\phi)F(\mathbf{R}_1,\mathbf{R}_2), \tag{B7}$$

where ϕ is the angle between $\mathbf{R}_1$ and $\mathbf{R}_2$ and the integration in (B7) is performed at given values of $|\mathbf{R}_1|=R_1$ and $|\mathbf{R}_2|=R_2$.

It is not very difficult to recognize that

$$\frac{\partial^2\Upsilon_0}{\partial r_1\partial r_2} = \frac{1}{r_1r_2}\int_0^{r_1} dR_1R_1\int_0^{r_2} dR_2R_2F_0(R_1,R_2).$$

With a logarithmic accuracy, it is reduced to

$$\frac{\partial^2\Upsilon_0}{\partial r_1\partial r_2} = \frac{1}{4}r_1r_2F_0(r_1,r_2). \tag{B8}$$

Comparing this expression with (B2), we conclude that the contribution to the correlation function (B2) associated with Υ_0 has the same logarithmic dependence as the correlation function F of the vorticity. The further analysis is based upon the fact that the functions F_m with $m\geq 1$ have one power of the logarithm less than F_0. This is the consequence of the definition (B7) since the main logarithmic contribution determined by the isotropic part of $F(\mathbf{R}_1,\mathbf{R}_2)$ is canceled at the integration over angles in (B7). Therefore, we should extract in the functions Υ_m ($m\geq 1$) figuring in the expansion (B6) only the terms that contain an additional logarithmic integration.

The function Υ_1 can be rewritten as

$$\Upsilon_1(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{2}\mathbf{r}_1\mathbf{r}_2\Big(\int_0^\infty dR_1\int_0^\infty dR_2F_1(R_1,R_2) + \int_0^{r_1} dR_1\int_0^{r_2} dR_2(1-R_1^2/r_1^2)(1-R_2^2/r_2^2)F_1(R_1,R_2)$$
$$-\int_0^\infty dR_1\int_0^{r_2} dR_2(1-R_2^2/r_2^2)F_1(R_1,R_2) - \int_0^\infty dR_2\int_0^{r_1} dR_1(1-R_1^2/r_1^2)F_1(R_1,R_2)\Big). \tag{B9}$$

The first term here does not contribute into $\langle\sigma_{\alpha\beta}\sigma_{\mu\nu}\rangle$ since the differentiation in (B2) kills the product $\mathbf{r}_1\mathbf{r}_2$, the second term does not contain a logarithmic integration and, consequently, can be omitted, the third and fourth terms in (B9) contain the logarithmic integration, but they will be killed by the differentiation in (B2) since they are linear in $\mathbf{r}_1$ or $\mathbf{r}_2$. Thus, we conclude that the term Υ_1 does not produce any relevant contribution into $\langle\sigma_{\alpha\beta}\sigma_{\mu\nu}\rangle$.

The next function Υ_2 can be written as

$$\Upsilon_2(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{8}\cos(2\varphi)\int_0^\infty dR_1R_1\int_0^\infty dR_2R_2 \times F_2(R_1,R_2)(u_1u_2)^2\,. \quad \text{(B10)}$$

Let us pass to the variables R,η, where $R_1 = R\eta$ and $R_2 = R/\eta$. Then the expression (B10) acquires the form

$$\Upsilon_2(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{4}\left[2(\mathbf{r}_1\mathbf{r}_2)^2 - r_1^2r_2^2\right]\int_0^\infty dR/R \times\int_0^\infty d\eta/\eta\left[1-\theta(r_1-R_1)(1-R_1^2/r_1^2)\right]^2 \times\left[1-\theta(r_2-R_2)(1-R_2^2/r_2^2)\right]^2 F_2(R\eta,R/\eta)\,, \quad \text{(B11)}$$

where θ is the step function. For a logarithmic function $F(\mathbf{R}_1,\mathbf{R}_2)$ the asymptotics of $F_2(R\eta,R/\eta)$ are $F_2\propto\eta^2$ at small η and $F_2\propto\eta^{-2}$ at large η. It ensures the convergence of the integral over η in (B11). The convergence of the integral over R in (B11) at small R is guaranteed by the presence of the prefactor at $F_2(R\eta,R/\eta)$, at R larger than r_1,r_2 this prefactor is equal to unity. Thus, we conclude that the integral in (B11) has a logarithmic character and, therefore, the term $\Upsilon_2(\mathbf{r}_1,\mathbf{r}_2)$ produces the relevant contribution into (B2) which should be taken into account in addition to the contribution associated with $\Upsilon_0(\mathbf{r}_1,\mathbf{r}_2)$.

As far as the contributions to (B2) associated with higher-order terms $\Upsilon_m(\mathbf{r}_1,\mathbf{r}_2)$ are concerned, they can be neglected with a logarithmic accuracy. The point is that the term $\Upsilon_m(\mathbf{r}_1,\mathbf{r}_2)$, in accordance with (B6), is determined by the integral containing the factor $(u_1u_2)^m$, which at $R_1,R_2\gg r_1,r_1$ is $(r_1r_2/R_1R_2)^m$, this dependence ensuring the convergence of the integrals over R_1 and R_2 for $m\geq 3$. It means that no additional logarithmic factors are produced at integrations giving $\Upsilon_m(\mathbf{r}_1,\mathbf{r}_2)$. We, thus, conclude that only the terms in (B2) associated with Υ_0 and Υ_2 should be taken into account with a logarithmic accuracy. That gives us the final answer following from (B8, B11)

$$\nabla_{1\gamma}\nabla_{1\beta}\nabla_{2\rho}\nabla_{2\nu}\Upsilon(\mathbf{r}_1,\mathbf{r}_2) \approx \frac{1}{4}\delta_{\alpha\beta}\delta_{\mu\nu}F_0(r_1,r_2) + \frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu} -\delta_{\alpha\beta}\delta_{\mu\nu})\int\frac{dR}{R}\int\frac{d\eta}{\eta}F_2\left(R\eta,\frac{R}{\eta}\right)\,. \quad \text{(B12)}$$

Here, the integral over R should be cut off from below on the scale determined by r_1,r_2 and from above on the external scale L. Taking into account (B2), we conclude that the first term in (B12) reproduces (in the main logarithmic approximation) the pair correlation function of the vorticity $\omega=\epsilon_{\alpha\beta}\nabla_\alpha v_\beta$ and the second term in (B12) gives us the leading contribution to the pair correlation function of the strain $s_{\alpha\beta}=1/2(\nabla_\alpha v_\beta+\nabla_\beta v_\alpha)$.

Consider now the particular case

$$F(\mathbf{R}_1,\mathbf{R}_2) = g\left(\ln\frac{L}{|\,\mathbf{R}_1-\mathbf{R}_2\,|}\right)\,, \quad \text{(B13)}$$

corresponding to the simultaneous correlation function. Then at calculating (B7), we can believe that

$$F(\mathbf{R}_1,\mathbf{R}_2)\simeq g(\xi)+g'(\xi)\ln\frac{\max(R_1,R_2)}{|\,\mathbf{R}_1-\mathbf{R}_2\,|}\,, \quad \text{(B14)}$$

where $\xi=\ln[L/\max(R_1,R_2)]$. Substituting (B14) into (B7) and using (B4), we find $F_0=g(\xi)$ and

$$F_2(R_1,R_2)\approx g'(\xi)\times\begin{cases}\frac{1}{2}\left(\frac{R_1}{R_2}\right)^2 & \text{if } R_1<R_2\\ \frac{1}{2}\left(\frac{R_2}{R_1}\right)^2 & \text{if } R_1>R_2\end{cases}\,. \quad \text{(B15)}$$

To calculate Υ_2 corresponding to (B15), it is worthwhile to return to (B10). The integral in (B10) will be determined by the region $R_1\simeq R_2$ (at large enough R_1,R_2) and, therefore, ξ can be substituted by $\ln(L/R_1)$. After this substitution the integral over R_2 can be taken explicitly which gives

$$\Upsilon(\mathbf{r}_1,\mathbf{r}_2)\approx\frac{1}{16}\left[2(\mathbf{r}_1\mathbf{r}_2)^2-r_1^2r_2^2\right]\int d\xi g'(\xi)\,. \quad \text{(B16)}$$

The upper limit in this integral can be estimated as $\ln[L/\max(r_1,r_2)]$. Then with the logarithmic accuracy $\int d\xi g'(\xi)=F_0$ and the substitution of (B16) into (B2) gives,

$$\langle\sigma_{\alpha\beta}(\mathbf{r}_1)\sigma_{\mu\nu}(\mathbf{r}_2)\rangle \simeq\frac{1}{8}(3\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu}-\delta_{\alpha\beta}\delta_{\mu\nu})F_0(r_1,r_2)\,, \quad \text{(B17)}$$

where we have added also the term originating from Υ_0. We see that the simultaneous pair correlation functions, both of the vorticity and of the strain, are determined by the single function F_0.

To prove that the correlation time of the strain is of the same order as the vorticity correlation time we consider the different-time pair correlation function $\langle s_{\alpha\beta}(t_1)s_{\mu\nu}(t_2)\rangle$ and show that it decays with t_1-t_2 by the same law as the pair vorticity correlation function. We assume $t_1>t_2$. As in the previous Appendix, we consider the variation of the strain correlator under the pumping variation and show that it is of the same order as the variation of $\langle\omega(t_1)\omega(t_2)\rangle$ for any time difference t_1-t_2. Our proof will be based on the explicit expression (3.5) for this variation. It is obvious from (3.4, 3.8) that for the time difference $|t_1-t_2|\lesssim\sigma^{-1}$

(where σ is the characteristic value of $\nabla_\alpha v_\beta$) the function $\delta F(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2)$ does not differ essentially from its simultaneous value. Therefore, the above reasons concerning the coincidence of the simultaneous correlation functions are valid in this case. Further, we will treat the case of a large time difference $t_1 - t_2 \gg \sigma^{-1}$, all the more we expect the correlation time to be large.

As was explained in the main text, to find $\boldsymbol{\rho}$ in (3.5), we should first solve the Eq. (3.4) in the time interval $t_2 < t < t_2$ with the initial condition $\boldsymbol{\varrho}_1(t_1, t_1) = \mathbf{r}_1$. The formal solution of this equation is $\boldsymbol{\varrho}_1(t_1, t) = \hat{w}(t_1, t)\mathbf{r}_1$, where $\hat{w}$ is the matrix which can be written in the form of the antichronological exponent:

$$\hat{w}(t_1, t) = \tilde{\mathrm{T}} \exp\left(-\int_t^{t_1} dt' \hat{\sigma}(t')\right) . \tag{B18}$$

Incompressibility condition makes $\hat{\sigma}$ traceless so that the antichronological exponent in (B18) has the unitary determinant. A 2×2 matrix with the unitary determinant can be written as

$$w_{\alpha\beta} = \alpha n_\alpha n_\beta + \beta n_\alpha m_\beta + \alpha^{-1} m_\alpha m_\beta , \tag{B19}$$

where $\mathbf{n}$ and $\mathbf{m}$ are orthogonal unit vectors and $\alpha > 0$. Actually at $t_1 - t_2 \gg \sigma^{-1}$, we have $\alpha \gg 1$ so that the last term in the expression (B19) for $w_{\alpha\beta}$ can be neglected.

If $t_1 - t_2 \ll \tau$, then the matrix $\hat{\sigma}(t)$ in r.h.s. of (3.4) can be regarded as constant. It means that the directions of the main axes $\mathbf{n}$, $\mathbf{m}$ do not depend on $t_1 - t_2$ and α grows exponentially with $t_1 - t_2$, the coefficient β in (B19) grows proportional to α (it can be checked directly). The same is true up to the values $t_1 - t \lesssim \tau$. Then the argument $\boldsymbol{\rho}$ in (3.5) is $(\alpha_1 n_\alpha n_\beta + \beta_1 n_\alpha m_\beta) r_{1\beta} - (\alpha_2 n_\alpha n_\beta + \beta_2 n_\alpha m_\beta) r_{2\beta}$, which can be rewritten in the form $(\alpha_2 n_\alpha n_\beta + \beta_2 n_\alpha m_\beta)(\gamma r_{1\beta} - r_{2\beta})$. Here, γ is some (exponential) function of $t_1 - t_2$. Therefore, we encounter the situation when the correlation function is expressed in terms of the difference $\gamma r_{1\beta} - r_{2\beta}$. Applying the same arguments as for the simultaneous correlation function we again can prove that $\delta\langle s_{\alpha\beta} s_{\mu\nu}\rangle$ is of order of δF. Physically it is quite natural since we are inside the correlation time of $s_{\alpha\beta}$ and this correlation is reproduced for $\delta\langle s_{\alpha\beta} s_{\mu\nu}\rangle$.

The vorticity correlation time is of the order of the transfer time τ_*. To show that the strain correlation time τ is of the same order, we assume for a while that $\tau \ll \tau_*$ and show that this leads to a contradiction. Consider now the case $\tau_* \ll t_1 - t_2 \gg \tau$. Then the statistics of σ determining the matrix $\hat{w}(t_1, t_2)$ is independent of the statistics of $\sigma(t)$ determining the evolution of the vector $\boldsymbol{\rho}$ in (3.8) for $t < t_2$. Therefore, the averaging over $\hat{\sigma}$ in (3.5) can be done in two steps: First we average over the statistics of $\sigma(t)$ for $t < t_2$ and then over the statistics of σ determining the matrix $\hat{w}(t_1, t_2)$. The result of the first averaging is simply the variation of the simultaneous correlation function $\delta F(\Delta t = 0)$ but with the space argument depending on the matrix $\hat{w}(t_1, t_2)$, which gives the variation (3.5) at the second averaging:

$$\delta F(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2) = \left\langle \delta F\left[\Delta t = 0, |\hat{w}(t_1, t_2)\mathbf{r}_1 - \mathbf{r}_2|\right]\right\rangle . \tag{B20}$$

In accordance with (B19), averaging in (B20) is averaging over the statistics of α and β and also averaging over the angle ϑ characterizing the direction of $\mathbf{n}$. Because of the isotropy of the statistics, this averaging is reduced to the integration over ϑ.

This integration can be performed explicitly if one takes into account that $\delta F(\Delta t = 0)$ is a logarithmic function of the space argument. The logarithm $\xi = \ln\left[L/|w(t_1, t_2)\mathbf{r}_1 - \mathbf{r}_2|\right]$ can be divided into two parts: $\xi = \xi_1 + \xi_2 = \ln(L/r) + \ln\left[r/|w(t_1, t_2)\mathbf{r}_1 - \mathbf{r}_2|\right]$, the first being the larger and ϑ independent and the second one being ϑ dependent and of an order of unity at an appropriate choice of the scale r, namely, at $r = \max(r_2, \alpha r_1)$. Therefore, the function $g(\xi)$ can be expanded as $g(\xi_1) + g'(\xi_1)\xi_2$. We see that averaging g over angles is reduced to averaging $\xi_2 = \ln(r_2/p)$, where $p^2 = r_2^2 - 2\alpha r_1 r_2 \cos(\vartheta - \varphi_2)\cos(\vartheta - \varphi_1) - 2\beta r_1 r_2 \cos(\vartheta - \varphi_2)\sin(\vartheta - \varphi_1) + \alpha^2 r_1^2 \cos^2(\vartheta - \varphi_1) + \alpha^2 r_1^2 \sin(\vartheta - \varphi_1)$, and φ_1 is the angle between $\mathbf{r}_1$ and X-axis, φ_2 is the angle between $\mathbf{r}_2$ and X axis and ϑ is the angle between $\mathbf{n}$ and X axis. The integral $\int d\vartheta \xi_2$ can be found explicitly using (B4). For brevity, we give here the result for the case $\beta = 0$ (since for $\beta \neq 0$ it will be qualitatively the same): $\int \xi_2 d\vartheta/2\pi = \ln(r/R)$, where

$$R = \max\left\{\sqrt{r_2^2 - \alpha r_1 r_2 \cos(\varphi_1 - \varphi_2) + \alpha^2 r_1^2/4}, \alpha r_1/2\right\} . \tag{B21}$$

We have seen in (B12) that the pair correlation function of the vorticity ω is determined by the zero Fourier harmonic of F that is in our case the variation of this correlation function is $\delta F_0 = \langle \delta F(\Delta t = 0, r)\rangle$, where averaging is performed over α, β. The pair correlation function of the vorticity is determined by the second Fourier harmonic of F that is in our case the variation of this correlation function is

$$\delta F_2 = \left\langle \frac{\partial}{\partial \xi_1} \delta F(\Delta t = 0, r) \times \int \frac{d\varphi_1}{2\pi} \ln(r/R) \exp(2i\varphi_2 - 2i\varphi_1) \right\rangle . \tag{B22}$$

The integral over angles can be estimated here as $\min(r_2/\alpha r_1, \alpha r_1/r_2)$. Taking this into account, we find after substitution of (B22) into (B12) the estimate $\langle \delta F(\Delta t = 0, r)\rangle$ for the second term on r.h.s. of (B12) [one integration in this term is a logarithmic one, this integration converts the derivative of $\delta F(\Delta t = 0)$ over ξ_1 into $\delta F(\Delta t = 0)$ itself]. Therefore, both terms in the r.h.s. of (B12) are of the same order. Actually the same assertion is valid for variations of all correlation functions.

Thus, we come to the contradiction. We suggested that the correlation time τ of the strain is less than the time τ_* of the spectral transfer and found that it leads to the conclusion that the correlation functions of the strain are of the same order as that of the correlation functions of the vorticity, which have the correlation time τ_*. It means that the correlation functions of the strain also have the correlation time τ_*. Formally, it is revealed in the fact that for $\tau = \tau_*$, we cannot treat the time difference $t_1 - t_2 \gg \tau$ since for $t_1 - t_2 \gg \tau_*$ our approach does not work. Above, we have spoken about correlations in time, but actually our assumption concerns also the correlation in space: the vorticity of different scales is strongly correlated.

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