Nonuniversality of the Scaling Exponents of a Passive Scalar Convected by a Random Flow

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We consider a passive scalar convected by a multiscale random velocity field with short yet finite temporal correlations. Taking Kraichnan's limit of a white Gaussian velocity as a zero approximation we develop the perturbation theory with respect to a small correlation time and small non-Gaussianity of the velocity. We derive the renormalization (due to temporal correlations and non-Gaussianity) of the operator of turbulent diffusion. That allows us to calculate the respective corrections to the anomalous scaling exponents of the scalar field and show that they continuously depend on velocity correlation time and the degree of non-Gaussianity. The scalar exponents are thus nonuniversal as was predicted by Shraiman and Siggia on a phenomenological ground. [S0031-9007(96)00157-3]

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The most striking feature of turbulence is its intermittent spatial and temporal behavior. Statistically, intermittency means substantial non-Gaussianity. For developed turbulence, where the correlation functions are scale invariant at the inertial interval of scales, the intermittency is manifested as an anomalous scaling of correlation functions. That means that some random field $\theta(\mathbf{r},t)$ has the structure functions $S_{2n} = \langle[\theta(t,\mathbf{r}_1) - \theta(t,\mathbf{r}_2)]^{2n}\rangle \propto r_{12}^{\zeta_{2n}}$ with the exponents ζ_{2n} that are not equal to $n\zeta_2$. As a result, the degree of non-Gaussianity, which may be characterized by the ratio S_{2n}/S_2^n, depends on the scale. Experiments and simulations show that the anomalous scaling of the scalar field passively convected by a fluid is much more strongly pronounced than the anomalous scaling of the velocity field itself [1–3]. It is in the problem of a passive scalar where consistent analytic theory of an anomalous scaling starts to appear [3–7]. It is intuitively clear that the physical reason for scalar intermittency is a spatial inhomogeneity of the advecting velocity. The analysis of the velocity field with smooth inhomogeneity shows, however, that there is no anomalous scaling of the scalar (actually, no scaling at all since all the correlation functions are logarithmic) whatever be the (finite) temporal correlations of the velocity [4,8,9]. Analytic treatment of a nonsmooth velocity was possible hitherto only in the scale-invariant case for the so-called Kraichnan's problem of a white advected scalar [4] where the correlation functions satisfy closed linear equations of the second order [8]. It has been shown [5,6] that, even without any temporal correlations, spatial nonsmoothness of the velocity provides for an anomalous scaling of the scalar. The anomalous parts appeared as zero modes of the operator of turbulent diffusion and entered the correlation functions due to matching conditions at the pumping scale [5–7]. The coefficients at the modes were thus pumping dependent while the form of any zero mode was universal, i.e., determined only by the exponent of the velocity spectrum and space dimensionality. In particular, the exponents ζ_n of the scalar were universal for the delta-correlated velocity.

Now, what is the role of velocity temporal behavior in building up intermittency of the scalar field? It was argued phenomenologically by Shraiman and Siggia [7] that the exponents of the scalar field depend on more details of the velocity statistics "than just exponents." Here we consider the simplest possible generalization of Kraichnan's problem and consistently derive the equations for scalar correlation functions in the case of short yet finite velocity correlation time τ_r, which is supposed to be a power function of the scale τ. The behavior of the ratio τ_r/t_r is important, where t_r is the turnover time at the scale r. If the ratio tends to zero at decreasing r then we return to the δ-correlated case. If the ratio increases at decreasing r, we encounter the problem of the quenched disorder type, which should be considered separately. We consider the marginal case of a complete self-similarity where $\epsilon = \tau_r/t_r$ does not depend on r and formulate the perturbation theory regarding the ratio as the small parameter of our theory. We show that ζ_2 does not depend on ϵ while ζ_n for $n > 2$ are ϵ dependent; that is, the set of the exponents is nonuniversal along with the prediction of [7]. The principal difference between the second and higher correlation functions is naturally explained in the language of zero modes: There is no zero mode (except constant) for the pair correlator while the zero modes of the high correlators depend on the precise form of the operator of turbulent diffusion which is ϵ dependent. This is formally similar to what has been discovered by Kadanoff, Wegner, and Polyakov in the theory of phase transitions: The critical exponents continuously depend on the amplitude of the operator term with dimension d added to the Hamiltonian [10,11].

Note that the results below cannot be directly applied to the description of scalar advection by a Kolmogorov turbulence: Because of the sweeping effect, the different-time velocity statistics is not scale invariant in the

Eulerian frame [12]. Our use of a scale-invariant velocity is intended to establish the general fact of the sensitive dependence of scalar exponents on the velocity statistics.

The advection of passive scalar $\theta(t,\mathbf{r})$ by an incompressible flow is governed by the equations

$$(\partial_t - \hat{P})\theta = \phi, \qquad \hat{P}(t) = -v^\alpha\nabla^\alpha + \kappa\nabla^2, \qquad \nabla^\alpha v^\alpha = 0, \tag{1}$$

where κ is the coefficient of molecular diffusion. The advecting velocity $\mathbf{v}$ and the source ϕ are independent random functions. A formal solution of (1) is

$$\theta(t,\mathbf{r}) = \int_{-\infty}^{t} dt_1 T\exp\left(\int_{t_1}^{t} dt'\,\hat{P}(t')\right)\phi(t_1,\mathbf{r}), \tag{2}$$

where $T\exp$ designates the chronologically ordered exponent. From (2) it follows

$$F_n(t,\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) \equiv \langle\theta(t,\mathbf{r}_1)\cdots\theta(t,\mathbf{r}_{2n})\rangle = \int_{-\infty}^{t} dt_1\cdots\int_{-\infty}^{t} dt_{2n}\,\hat{\mathcal{A}}\left\langle\prod_{i=1}^{2n}\phi(t_i,\mathbf{r}_i)\right\rangle, \tag{3}$$

$$\hat{\mathcal{A}} = \langle\hat{Q}\rangle, \qquad \hat{Q}(t) = \prod_{i=1}^{2n} T\exp\left(\int_{t_i}^{t} dt_i'\hat{P}(t_i',\mathbf{r}_i)\right). \tag{4}$$

Differentiating $\hat{\mathcal{A}}$ over the current time t, one gets

$$\partial_t\hat{\mathcal{A}} = \langle\hat{\mathcal{P}}(t)\hat{Q}(t)\rangle, \quad \hat{\mathcal{P}} = \sum_i(-v_i^\alpha\nabla_i^\alpha + \kappa\nabla_i^2), \tag{5}$$

where $\mathbf{v}_i = \mathbf{v}(t,\mathbf{r}_i)$ and $\nabla_i = \partial/\partial\mathbf{r}_i$. Below summation over both repeated vector superscripts and subscripts enumerating points $\mathbf{r}_i$ is implied. The identity (5) can be brought to the form

$$\partial_t\hat{\mathcal{A}}(t) = \kappa\nabla_i^2\hat{\mathcal{A}}(t) + \int_0^\infty dt'\,\hat{\mathcal{N}}(t')\hat{\mathcal{A}}(t-t'), \tag{6}$$

where $\hat{\mathcal{N}}(t)$ is to be found. The decay of $\hat{\mathcal{N}}(t)$ is determined by the velocity correlation time τ_r which is supposed to be much smaller than the spectral transfer time characteristic of (3). It is the reason why the upper limit in (6) can be substituted by infinity.

We shall find the first terms of the expansion of $\hat{\mathcal{N}}$ in τ_r. Let us first examine the Gaussian contribution to $\partial_t\hat{\mathcal{A}}$ related to reducible correlation functions of $\mathbf{v}$. It is well known that at Gaussian averaging $\langle xf(x)\rangle = \langle x^2\rangle\langle\partial f/\partial x\rangle$. Generalizing the trick for the case of the operator product we obtain

$$\int^{t} d\tilde{t}\,\underline{v}_i^\alpha(t)\nabla_i^\alpha\left\langle T\exp\left[\int_{\tilde{t}}^{t} dt'\hat{\mathcal{P}}(t')\right]\underline{v}_j^\beta(\tilde{t})\nabla_j^\beta\hat{Q}(\tilde{t})\right\rangle_G, \tag{7}$$

where the product $\underline{v}_i^\alpha(t)\underline{v}_j^\beta(\tilde{t})$ should be substituted by the pair correlation function $\langle v^\alpha(t,\mathbf{r}_i)v^\beta(\tilde{t},\mathbf{r}_j)\rangle$. The integrand of (7) is nonzero for $t-\tilde{t} \le \tau_r$ and consequently $T\exp[\int_{\tilde{t}}^{t} dt'\,\hat{\mathcal{P}}(t')]$ can be expanded over $t-\tilde{t}$. The zero term gives

$$\hat{\mathcal{N}}_0(t) = \langle v_i^\alpha(t)\nabla_i^\alpha v_j^\beta(0)\nabla_j^\beta\rangle. \tag{8}$$

The first and second terms of the expansion produce the linear in τ_r contribution

$$\begin{aligned}\hat{\mathcal{N}}_1(t) = \int_0^t dt_1\Big[\int_0^{t_1} dt_2[\underline{v}_i^\alpha(t)\nabla_i^\alpha\overline{v}_k^\gamma(t_1)\nabla_k^\gamma\underline{v}_j^\beta(t_2)\\ \times\nabla_j^\beta\overline{v}_m^\mu(0)\nabla_m^\mu + \underline{v}_i^\alpha(t)\nabla_i^\alpha\overline{v}_k^\gamma(t_1)\\ \times\nabla_k^\gamma\overline{v}_m^\mu(t_2)\nabla_m^\mu\underline{v}_j^\beta(0)\nabla_j^\beta]\\ + \kappa t_1\underline{v}_i^\alpha(t_1)\nabla_i^\alpha\nabla_k^2\underline{v}_j^\beta(0)\nabla_j^\beta\Big],\end{aligned} \tag{9}$$

where the products $\overline{v}\,\overline{v}$ and $\underline{v}\,\underline{v}$ should be substituted by the corresponding pair correlation functions.

For a short-correlated velocity field, the leading non-Gaussian contribution to the correlation functions of $\mathbf{v}$ is determined by the irreducible part of the fourth-order correlation function of $\mathbf{v}$. Generalizing the trick leading from (5) to (7) we obtain the non-Gaussian term $\hat{\mathcal{N}}_{\rm nG}(t)$:

$$\int_0^t dt_1\int_0^{t_1} dt_2\,\langle\!\langle v_i^\alpha(t)\nabla_i^\alpha v_k^\mu(t_1)\nabla_k^\mu v_j^\beta(t_2)\nabla_j^\beta v_n^\gamma(0)\nabla_n^\gamma\rangle\!\rangle, \tag{10}$$

where double angular brackets stand for the cumulant.

The operator $\hat{\mathcal{A}}(t)$ is exponential in time

$$\hat{\mathcal{A}}(t) = \exp[(t-t_0)(\kappa\nabla_i^2 + \hat{\mathcal{L}})]\hat{\mathcal{A}}(t_0) \tag{11}$$

asymptotically at $t - t_0 \gg \tau_r$. Substituting (11) into (6), expanding $\exp(t'\hat{\mathcal{L}})$, and keeping only the principal terms we find the operator of turbulent diffusion

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_1' + \hat{\mathcal{L}}_{\rm nG}, \tag{12}$$

$$\hat{\mathcal{L}}_{\{0,1,\rm nG\}} = \int_0^\infty dt\,\hat{\mathcal{N}}_{\{0,1,\rm nG\}}(t),$$

$$\hat{\mathcal{L}}_1' = -\int_0^\infty dt\,t\mathcal{N}_0(t)\hat{\mathcal{L}}_0.$$

Using (3) and (11) we obtain an expression for $\partial_t F_n$. For the pumping δ correlated in time, one gets

$$\partial_t F_n(t,\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) - \hat{\mathcal{L}}F_n(t,\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) = \hat{\mathcal{M}}[\chi_{12}F_{n-1}(t,\mathbf{r}_3,\ldots,\mathbf{r}_{2n}) + \text{permutations}]. \tag{13}$$

Here the function $\chi(r_{12}) = \int dt\,\langle\phi(t,\mathbf{r}_1)\phi(0,\mathbf{r}_2)\rangle$ decays on the pumping scale L and $\chi(0)$ is the production rate of θ^2. The operator $\hat{\mathcal{M}}$ in (13) can be estimated as $\hat{\mathcal{A}}(\tau_L)$. The account of temporal correlations of the pumping (which can be done perturbatively as long as the pumping correlation time is much less than the time of scalar transfer) results in extra renormalization of the $\hat{\mathcal{M}}$ operator. Its explicit form is unimportant for what follows. Indeed, the balance equation (13) contains the renormalization (due to velocity temporal correlations and non-Gaussianity) of all three relevant quantities: pumping, turbulent diffusion, and molecular diffusion (the last term in $\hat{\mathcal{N}}_1$). We discuss here only the scaling exponents in the convective interval of scales (see below) that are determined solely by the form of the operator of turbulent diffusion $\hat{\mathcal{L}}$.

Let us consider the pair correlation function of the velocity to be scale invariant:

$$\langle[v^\alpha(t,\mathbf{r}) - v^\alpha(0,\mathbf{0})][v^\beta(t,\mathbf{r}) - v^\beta(0,\mathbf{0})]\rangle = 2K^{\alpha\beta}(t,r),$$

$$K^{\alpha\beta} = \frac{Dr^{2-\gamma}}{\tau^r}\left[\left(\delta^{\alpha\beta} - \frac{r^\alpha r^\beta}{r^2}\right)g_\perp\left(\frac{|t|}{\tau_r}\right) + \delta^{\alpha\beta}g_\parallel\left(\frac{|t|}{\tau_r}\right)\right]$$

with the correlation time $\tau_r = \tau_L(r/L)^z$. Dimensionless functions $g_\perp$ and $g_\parallel$ satisfy the incompressibility condition $(d-1)g_\perp(x) = zx^a d[x^{1-a}g_\parallel(x)]/dx$ where $a = (2-\gamma)/z - 1$. Their normalization is fixed by expressions (15) and (18) below. The main term in (12) is [8]

$$\hat{\mathcal{L}}_0 = -\sum_{ij}\mathcal{K}_0^{\alpha\beta}(r_{ij})\nabla_i^\alpha\nabla_j^\beta,$$

$$\mathcal{K}_0^{\alpha\beta} = 2\int_0^\infty dt\,\mathcal{K}^{\alpha\beta}(t), \tag{14}$$

$$\mathcal{K}_0^{\alpha\beta} = Dr^{2-\gamma}\left(\frac{d+1-\gamma}{2-\gamma}\delta^{\alpha\beta} - \frac{r^\alpha r^\beta}{r^2}\right). \tag{15}$$

Expressions (14) and (15) lead to the turnover time $t_r = (2-\gamma)r^\gamma/D\gamma d(d-1)$ obtained for the delta-correlated case [4,6]. Our marginal case corresponds to $z = \gamma$ and the small parameter of the perturbation theory is thus

$$\epsilon = D\tau_L L^{-\gamma}\gamma d(d-1)/(2-\gamma) \ll 1. \tag{16}$$

Note that ϵ contains d^2, which tells us that the space dimensionality should not be very large for the approximation of a short correlation to be valid: The characteristic time of the scalar transfer (proportional to d^{-2}) should be larger than the correlation time.

Starting from the expression for the pair velocity correlator we can obtain the first Gaussian ϵ correction to (14) by calculating (9) and then integrals in (12)

$$\hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_1' = \frac{1}{2}\sum_{i,j,k}K_{0;ij}^{\alpha\beta}K_{1;ik}^{\mu\nu;\alpha}\nabla_i^\mu\nabla_j^\beta\nabla_k^\nu - \frac{1}{2}\sum_{i,j}\mathcal{B}_{ij}^{\mu\nu}\nabla_i^\mu\nabla_j^\nu - \frac{\kappa}{2}\sum_{i,j,k}\nabla_k^2\mathcal{K}_{1;ij}^{\alpha\beta}\nabla_i^\alpha\nabla_j^\beta,$$

$$\mathcal{B}^{\mu\nu}(\mathbf{r}) = K_{1;ij}^{\alpha\mu;\beta}K_{0;ij}^{\nu\beta;\alpha} - K_{1;ij}^{\alpha\beta}K_{0;ij}^{\mu\nu;\alpha\beta} + 2\int_0^\infty dt_1\,\nabla_\mathbf{r}^\alpha\nabla_\mathbf{r}^\beta\left[\int_{t_1}^\infty dt_2\,K^{\alpha\beta}(t_2;\mathbf{r})\int_{t_1}^\infty dt_3\,K^{\mu\nu}(t_3;\mathbf{r}) - \int_{t_1}^\infty dt_2\,K^{\alpha\mu}(t_2;\mathbf{r})\int_{t_1}^\infty dt_3\,K^{\beta\nu}(t_3;\mathbf{r})\right], \tag{17}$$

where $K_n^{\alpha\beta;\mu} \equiv \nabla_\mathbf{r}^\mu K_n^{\alpha\beta}$, $K_n^{\alpha\beta;\mu\nu} \equiv \nabla_\mathrm{r}^\mu\nabla_\mathbf{r}^\nu K_n^{\alpha\beta}$,and

$$K_1^{\alpha\beta}(\mathbf{r}) = 2\int_0^\infty dt\,K^{\alpha\beta}(t,\mathbf{r}) = Dr^{2-\gamma}\tau_r\left(\frac{d+1}{2}\delta^{\alpha\beta} - \frac{r^\alpha r^\beta}{r^2}\right),$$

$$\mathcal{B}^{\alpha\beta}(\mathbf{r}) = \epsilon Dr^{2-\gamma}\left[b_\parallel\delta^{\alpha\beta} + b_\perp\left(\delta^{\alpha\beta} - \frac{r^\alpha r^\beta}{r^2}\right)\right]. \tag{18}$$

Now we can analyze Eq. (13) for F_n. At the convective interval of scales $L \gg r \gg [\kappa(2-\gamma)/D(d-1)]^{1/(2-\gamma)}$, the molecular diffusion term can be dropped: It is enough to require $F_n = 0$ at $r = 0$ [5,6]. Here, the zero modes of $\hat{\mathcal{L}}$ are responsible for the anomalous scaling of F_n. The scaling exponents of the bare operator $\hat{\mathcal{L}}_0$ and the perturbation operator $\hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_1'$ coincide. For self-similar velocity statistics, the non-Gaussian contribution $\hat{\mathcal{L}}_{\mathrm{nG}}$ has the same scaling too. The first consequence is that the exponent of the pair correlation function is γ at arbitrary finite order in ϵ for any γ and d. Indeed, there is no zero mode of the two-point $\hat{\mathcal{L}}$ with a nonzero positive exponent that could provide an anomaly. Contrary, for $n > 2$ the account of the ϵ contributions to the bare operator $\hat{\mathcal{L}}_0$ should produce obviously ϵ-dependent corrections to the exponents of zero modes and consequently ϵ-dependent anomalous scaling. This is similar to exponent nonuniversality due to marginal variables in renormalization group [10,11,13].

To illustrate the above conclusion about the τ dependence of the scalar exponents, let us give an example where the calculation can be done explicitly. We consider a large dimensionality [the limit $\gamma \gg (2-\gamma)/d$ solved in [6,14] for $\tau = 0$] while assuming that, in addition to (16), $1/d \gg \epsilon$ (it will be seen below how the parameter γ enters the condition). The leading (in d) terms of the bare and Gaussian perturbative operators in terms of relative distances r_{ij} are as follows [multiplied by $(2-\gamma)/dD$]:

$$\hat{\mathcal{L}}_{0,0} = d\sum_{i>j} r_{ij}^{1-\gamma}\partial_{r_{ij}}, \qquad \hat{\mathcal{L}}_{0,1} = \sum_{i>j} r_{ij}^{1-\gamma}(r_{ij}\partial_{ij}^2 - \gamma\partial_{ij}) - \frac{1}{2}\sum_{i,j,p,q} r_{ij}^{2-\gamma}\frac{\mathbf{r}_{ip}\mathbf{r}_{jq}}{r_{ip}r_{jq}}\partial_{ip}\partial_{jq}\,,$$

$$\hat{\mathcal{L}}_{1,0} = \frac{\epsilon d(2-\gamma)}{\gamma}\left[\sum_{k>l} r_{kl}\partial_{kl} + \gamma - 1 + 2b_{\parallel}^{(0)}\right]\sum_{i>j} r_{ij}^{1-\gamma}\partial_{ij}\,,$$

$$\hat{\mathcal{L}}_{1,1} = \frac{\epsilon(\gamma-2)}{8\gamma}\Bigg[\sum\left(\frac{\mathbf{r}_{ip}\mathbf{r}_{jp}}{r_{ip}r_{jq}}r_{ij}^2 r_{kl}^{1-\gamma}\partial_{ip}\partial_{jp}\partial_{kl} + \frac{\mathbf{r}_{kp}\mathbf{r}_{lq}}{r_{kp}r_{lq}}r_{ij}r_{kl}^{2-\gamma}\partial_{kp}\partial_{lq}\partial_{ij}\right)$$
$$+ 4\sum\frac{\mathbf{r}_{jp}\mathbf{r}_{ik}}{r_{ik}r_{jp}}\left(r_{ij}^{2-\gamma} + \frac{2-\gamma}{2}r_{ij}^2 r_{ik}^{-\gamma}\right)\partial_{jp}\partial_{ik} + 4b_{\parallel}^{(0)}\sum r_{ij}^{2-\gamma}\frac{\mathbf{r}_{ip}\mathbf{r}_{jq}}{r_{ip}r_{jq}} + 16\sum_{i>j,k>l} r_{ij}^{1-\gamma}r_{kl}\partial_{ij}\partial_{kl}$$
$$+ [16 - 4\gamma - 8(b_{\parallel}^{(1)} + b_{\perp}^{(0)} - b_{\parallel}^{(0)})]\sum_{i>j} r_{ij}^{1-\gamma}\partial_{ij}\Bigg]. \tag{19}$$

The summation is performed over $n(n-1)/2$ distances, which are independent variables if $d > n - 2$. For the chosen form of $K^{\alpha\beta}(t,\mathbf{r})$, $b_{\parallel} \to (2-\gamma)\times(b_{\parallel}^{(0)}d^3 + b_{\parallel}^{(1)}d^2)$, $b_{\perp} \to (2-\gamma)b_{\perp}^{(0)}d^2$ at $d\to\infty$, with d-independent constants $b_{\parallel,\perp}^{(i)}$. First, we calculate the corrections to the exponents related to (19) and then discuss the corrections due to non-Gaussianity.

Solving the equation for the pair correlation function one can check that $\zeta_2 = \gamma$ independent of ϵ and d. Then we consider the four-point correlation function. To get the main contribution at $r \ll L$ one has to perturb the bare zero mode of $\hat{\mathcal{L}}_{0,0} + \hat{\mathcal{L}}_{0,1}$. In the limit under consideration, it is enough to consider the mode

$$Z_0 = \sum_{\{i,j,k,l\}}(r_{ij}^{\gamma} - r_{kl}^{\gamma})^2 - 1/2\sum_{\{i,j,k\}}(r_{ij}^{\gamma} - r_{ik}^{\gamma})^2\,, \tag{20}$$

with the leading exponent $\Delta_4(0) = 4(2-\gamma)/d$ found in [6] by the $1/d$ expansion. The first ϵ correction to (20) can be obtained by applying the operator $-\hat{\mathcal{L}}_{0,0}^{-1}(\hat{\mathcal{L}}_{1,0}\hat{\mathcal{L}}_{0,0}^{-1}\hat{\mathcal{L}}_{0,1} + \hat{\mathcal{L}}_{0,1}\hat{\mathcal{L}}_{0,0}^{-1}\hat{\mathcal{L}}_{1,0})$ to Z_0. The correction to the exponent is determined by the coefficient at $\ln(L/r)$ in the first ϵ contribution to Z_0,

$$\Delta_4(\epsilon) = \Delta_4(0) + \frac{\epsilon(2-\gamma)}{d\gamma}(4 + 6\gamma - 2\gamma^2)\,. \tag{21}$$

We can also calculate τ-related corrections for the high-order functions by using the technique developed in [14] for finding the largest exponent. For $n \ll \gamma d$,

$$\Delta_{2n}(\epsilon) = n(n-1)\Delta_4(\epsilon)/2\,. \tag{22}$$

The scaling exponents thus depend not only on purely dimensionless quantities γ and d but also on a dimensionless ratio of dimensional quantities. In other words, the exponents depend on the form of the structural functions $g_{\perp}$ and $g_{\parallel}$ as well as on large-scale quantities via the dependence of the argument of the functions on L.

Considering the opposite hierarchy $1/d \ll \epsilon$ and neglecting $1/d$ corrections one finds $\zeta_{2n} = n\zeta_2$ at any order in ϵ: temporal correlations by themselves do not produce an anomalous scaling if it is absent in the uncorrelated case. In this limit, the anomalous exponents appear only in the next $1/d$ order and are proportional to ϵd.

Now let us discuss the non-Gaussian contribution. We denote by ϵ_4 the ratio of the cumulant to the fourth-order correlator and consider $1 \gg 1/d \gg \epsilon_4 d^3$. Using (10), (12), and (13) we conclude that the contribution to Δ_n is proportional to $n(n-1)\epsilon_4 d^2$ for $n \ll \gamma d$.

To conclude, we learned that the scalar exponents are sensitive to the details of the velocity statistics. The existence of two different contributions (due to temporal correlations and non-Gaussianity of the velocity) makes it possible that there exist some classes of the statistics with special relations between the contributions their analysis is left for future studies. Hopefully, real turbulent flows belong to those classes and analytic expressions for the scalar exponents can be found some day.

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