Counting Statistics of Charge Transport

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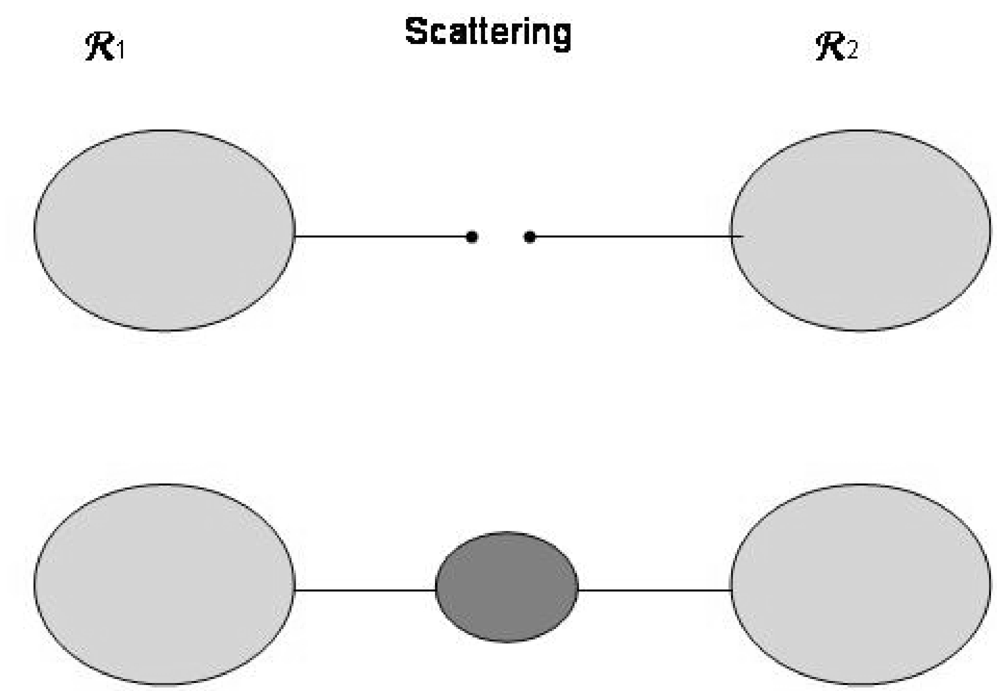
Outline:

- Counting Statistics - A simple setting.
- Full counting statistics.
- Convergence and Regularization:
  - ◇ Thermodynamic limit
  - ◇ Linear dispersion .
- Interpretation, Comparison between a continuous measurement of current and measurement of charge.
- Elaboration on the first moments.
- The many cycle limit: When is the pumping "extensive" in time?

Setting:

• The system consists of reservoirs: $\mathcal{R}_1$, $\mathcal{R}_2$, ...
The reservoirs are coupled at time $t = 0$ and decoupled at time T.



- Measured quantity: charge at the reservoirs in the end compared to the initial charge.

- For simplicity consider just the charge entering and leaving reservoir 1, and denote $\widehat{Q}$ the projection on $\mathcal{R}_1$.

- A problem of a quantum field coupled to "classical" controlled external potential. The setting applies also to other processes involving transfer of electrons.

- Full counting statistics was introduced in:

L.S. Levitov and G.B. Lesovik, (1993) *JETP Lett.*, **58**, pp. 230–235

D.A. Ivanov and L.S. Levitov, (1993) , *JETP Lett.*, **58**, pp. 461–468

D.A. Ivanov, H.W. Lee and L.S. Levitov, (1997) *Phys. Rev.*, B**56**, pp. 6839–6850

L.S. Levitov, H.W. Lee and G.B. Lesovik, (1996), *JMP*, **37**, pp. 4845–4866

Full counting statistics

- The statistics of charge transferred is described by derivatives of:

$$\chi(\lambda) = \sum P(\text{charge in } \mathcal{R}_1 \text{ changed by } n) e^{i\lambda n}$$

- For adiabatic change, and short scattering time Levitov and Lesovik, obtained the following expression for χ:

$$\chi(\lambda) = \det(1 + n(S^\dagger e^{iq\lambda} S e^{-iq\lambda} - 1)) \quad (1)$$

  Where n is the occupation number operator and S is the scattering matrix.

  It was remarked that **"this expression requires careful understanding and regularization"**.

- **Assume that:**
  - α, β are a basis for the Fock space which are **eigenstates** of the second quantized charge operator $\mathbb{Q}$ of reservoir 1.
  - $\mathbb{U}(T)$ is the evolution in Fock space
  - ρ is the initial density matrix, which is assumed **diagonal** in α.

  Then:

$$\chi(\lambda, T) = \sum_{\alpha,\beta} P(\alpha(t=0), \beta(t=T)) e^{i\lambda(Q[\beta]-Q[\alpha])} \quad (2)$$

$$= \sum_{\alpha,\beta} < \alpha|\rho|\alpha > | < \alpha|\mathbb{U}^\dagger|\beta > |^2 e^{i\lambda(Q[\beta]-Q[\alpha])}$$

$$= \mathsf{Tr}(\rho \mathbb{U}^\dagger(T) e^{i\lambda\mathbb{Q}} \mathbb{U}(T) e^{-i\lambda\mathbb{Q}})$$

Non interacting particles:

- For a single particle operator e^A define the second quantized operator

$$\Gamma(e^A) = exp(\sum A_{ij} a_i^\dagger a_j)$$

- $\Gamma(e^A)\Gamma(e^B) = \Gamma(e^A e^B)$ this can be verified by checking:

$$[A_{ij} a_i^\dagger a_j, B_{kl} a_k^\dagger a_l] = [A, B]_{mn} a_m^\dagger a_n \tag{3}$$

- For particles this reflects that:

$$\Gamma(e^A) = e^A \oplus (e^A \otimes e^A) \oplus (e^A \otimes e^A \otimes e^A) \oplus ...$$

  on the Fock space $\oplus_n Sym(Asym)(\otimes^n H)$

- For example, for **non interacting particles**, Bosons or Fermions, $\mathbb{U}$ is obtained from the single particle evolution U by:

$$\mathbb{U} = \Gamma(U) = U \oplus (U \otimes U) \oplus (U \otimes U \otimes U) \oplus ...$$

  It is evident that $\Gamma(U_1 U_2) = \Gamma(U_1)\Gamma(U_2)$

- We will use the following formula

$$\mathrm{Tr}(\Gamma(C)) = \prod_i (1-\xi e^{\mu_i})^{-\xi} = \det(1-\xi e^{C})^{-\xi} \qquad (4)$$

  Where $\xi = 1$ for bosons and $\xi = -1$ for fermions.

- This is just the partition function of non interacting particles, with Hamiltonian C/β.

- For non interacting particles:

$$\mathrm{Tr}(\Gamma(e^{A})\Gamma(e^{B})...) = \det(1-\xi e^{A}e^{B}...)^{-\xi} \qquad (5)$$

- All the operators appearing in (2) are of the form Γ(...) so:

$$\chi(\lambda, T) = \frac{1}{Z}\mathrm{Tr}(\Gamma(e^{-\beta H_0}U^\dagger e^{i\lambda\hat{Q}}Ue^{-i\lambda\hat{Q}})) \tag{6}$$

- Formally χ is similar to a partition function, and $\log\chi$ to a thermodynamic potential with respect to λ and the extensive parameter T

$$\chi(\lambda) = \tfrac{1}{Z}\,\mathrm{det}(1 + e^{-\beta H_0}(U^\dagger e^{i\lambda\hat{Q}}Ue^{-i\lambda\hat{Q}})) = \tag{7}$$
$$\mathrm{det}(1 + n(U^\dagger e^{i\lambda\hat{Q}}Ue^{-i\lambda\hat{Q}} - 1))$$

Where $Z = \mathrm{det}(1+e^{-\beta H_0})$ and n is the occupation number operator $\frac{e^{-\beta H_0}}{1+e^{-\beta H_0}}$ at the initial time (H_0 is the initial Hamiltonian).

- **The adiabatic limit:** $S = \lim_{t\to\infty} e^{iH_0t}U(t,-t)e^{iH_0t}$. Since $\hat{Q}$ commutes with H_0, one obtains in the limit of $T \to \infty$:

$$\chi(\lambda) = \mathrm{det}(1 + n(S^\dagger e^{i\lambda\hat{Q}}Se^{-i\lambda\hat{Q}} - 1)) \tag{8}$$

Convergence and Regularization

- A determinant of the form $\det(1+A)$ is well defined if the operator A has a well defined trace ($A \in \mathcal{J}_1$=*trace class*). Then

$$\log\det(1+A) = \mathrm{Tr}A + \frac{1}{2}\mathrm{Tr}A^2 + ...$$

.

What about the operator $n_d(U^\dagger e^{i\lambda\widehat{Q}} U e^{-i\lambda\widehat{Q}} - 1)$?

- Two basic problems: "IR", and "UV":
  - ◇ Thermodynamic limit - large system, $\log$ Vol charge fluctuations.
  - ◇ For linear dispersion - energy unbounded bellow.

Sketch of validity proof for quadratic dispersion.

- Show that

  $$n_d(U^\dagger e^{i\lambda\hat{Q}} U e^{-i\lambda\hat{Q}} - 1)$$

  has a well defined trace if

  $$n(U_0^\dagger e^{i\lambda\hat{Q}} U_0 e^{-i\lambda\hat{Q}} - 1)$$

  has a well defined trace.
  Where U_0 is free, connected evolution.

- Assume that the system is driven by a Hamiltonian $H(t) = p^2 + V(t)$ where $V(t)$ is a local potential supported at the pump.

- For quadratic dispersion there is finite density of particles.

- Note that if $A \in \mathcal{J}_1$ and B is a bounded operator then $AB \in \mathcal{J}_1$. In our case all of the operators appearing are bounded.

- Show that one can replace n_d by n (i.e. $(n - n_d) \in \mathcal{J}_1$) Avron et. al.

- **Birman-Solomyak** criterion: If A is diagonal in the p representation and B diagonal in x representation then $\mathrm{Tr}AB = \int \mathrm{d}pA(p) \int \mathrm{d}xB(x)$ if the integrals exist.

- $n(U(T) - U_0(T)) \in \mathcal{J}_1$ where $U_0(T) = e^{-ip^2T}$:

$$\mathrm{Tr}(|n(U - U_0)|) \leq \int_0^T ||nU_0(T-t)VU(t)||_1 \mathrm{d}t \leq \int_0^T \int |n(p)|\mathrm{d}p \int |V(x,t)|\mathrm{d}x\mathrm{d}t$$

- Thus the statement is equivalent to proving validity for free connected evolution.

- Last step: prove

$$n(U_0^\dagger e^{i\lambda\hat{Q}} U_0 e^{-i\lambda\hat{Q}} - 1) \in \mathcal{J}_1$$

All operators are well known, standard estimates.

Regularized determinant for the linear dispersion case.

- Note **particle - hole symmetry**: $(n, \lambda) \Rightarrow (1 - n, -\lambda)$,

$$\det(1 + (1-n)(e^{-i\lambda \hat{Q}_T} e^{+i\lambda \hat{Q}} - 1)) \tag{9}$$

  Where $\hat{Q}_T = U^\dagger \hat{Q} U$.

- This suggests to look for a formula that involves particles and holes:

- **Regularized formula** by subtracting and adding the first moment:

$$\chi(\lambda)_{reg} =$$
$$\det(1 + n(e^{i\lambda \hat{Q}_T} e^{-i\lambda(1-n)\hat{Q}} e^{-i\lambda n \hat{Q}_T} - 1) + (n-1)(e^{in\lambda \hat{Q}} e^{-i\lambda n \hat{Q}_T} - 1)) e^{i\lambda \mathrm{Tr}\{(UnU^\dagger - n)\hat{Q}\}}$$

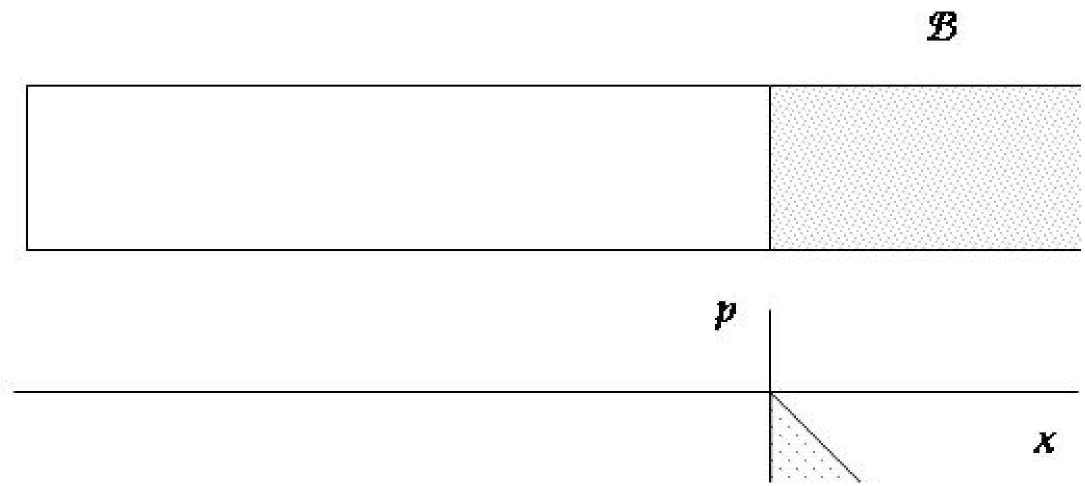
  – Note $e^{-i\lambda n \hat{Q}_T}$ is not unitary because $n\hat{Q}_T$ is not hermitian. this can be amended by taking instead $e^{-i\lambda n \hat{Q}_T n}$

  – An equivalent result, valid only at zero temperature appeared in B. A. Muzikantskii and Y. Adamov, cond-mat/0301075.

Interlude: Classical picture

• Classical particles in a box:

Number of particles leaving a box which is opened for a time t. Let p be the single particle probability of leaving.



$$\chi(\lambda) = \sum P(\text{n particles left the box})e^{i\lambda n}$$

Assume particles are statistically independent. if χ_1 is the characteristic function for a single particle ,

$$\chi(\lambda) = \prod_{\text{particles}} \chi_1(\lambda) = (q + e^{i\lambda} p)^N \tag{10}$$

Where N is the number of particles and $p + q = 1 \Rightarrow$
we get a **binomial distribution**.

Let $B \to \infty$, $N \to \infty$ and $N/B = n = const.$ (i.e. the density of particles remains const).

As we enlarge the box $p \to \frac{p}{B}$. thus

$$\chi(\lambda) = \lim_{N\to\infty}(1 - \frac{p}{B} + e^{i\lambda}\frac{p}{B})^N = \tag{11}$$
$$\lim_{N\to\infty}(1 + (e^{i\lambda} - 1)\frac{p}{N/n})^N = e^{(e^{i\lambda}-1)pn}$$

Which is a **poisson distribution**.

- In the quantum statistical mechanics world the picture is different:

  Example: occupy just the quantum state $|1>$ then $n = |1><1|$ and

$$\chi(\lambda) = \det(1 + |1><1|(\Lambda - 1)) = <1|\Lambda|1>$$

  Where $\Lambda = U^{\dagger} e^{i\lambda Q} U e^{-i\lambda Q}$

- if we occupy also $|2>$ then $n = |1><1| + |2><2|$ and

$$\chi(\lambda) = \det(1+n(\Lambda-1)) = \det \begin{pmatrix} <1|\Lambda|1> & <1|\Lambda|2> \\ <2|\Lambda|1> & <2|\Lambda|2> \end{pmatrix} \neq <1|\Lambda|1><2|\Lambda|2>$$

- Note however, usually for open systems decay of non-diagonal in time.

Direct Current Measurements:

- We start of with the wrong option:

$$\chi_{wrong}(\lambda) =< e^{i\lambda(\hat{Q}_T - \hat{Q})} >=< e^{i\lambda \int \hat{I}(t')\mathrm{d}t'} >$$

  $\hat{Q}_T - \hat{Q}$ is not a good quantum mechanical observable:
  Doesn't measure the state of the system but contains the future - you can't measure it again.
  While Q has integer spectrum, $\hat{Q}_T - \hat{Q}$ has continuous spectrum $\Rightarrow$ not a good measure of charge transfer.

- Measurement using an auxiliary **quantum mechanical detector** such as a spin or other device:

$$\chi_{detector}(\lambda) =< \overleftarrow{\mathcal{T}} e^{i\lambda/2 \int_0^t I(t')\mathrm{d}t'} \overrightarrow{\mathcal{T}} e^{i\lambda/2 \int_0^t I(t')\mathrm{d}t'} > \quad (12)$$

  Where $\mathcal{T}$ is time ordering. A general approach:
  Yu.V. Nazarov, and M. Kindermann,(2001), cond-mat/0107133
  Difference between statistics schemes:
  G. B. Lesovik and N. M. Shelkachev cond-mat/0303024 (in Russian!)

- Relation to the expression

$$\chi(\lambda) =< e^{i\lambda Q(T)} e^{-i\lambda Q(0)} > \tag{13}$$

Write in path integral language the same quantities:

$$\chi(\lambda, T) = \int_{\substack{\xi_1(0)=\xi_2(0) \\ \xi_2(T)=\xi_1(T)}} \mathcal{D}[\xi_1]\mathcal{D}[\xi_2]\rho(\xi_1(0), \xi_2(0)) e^{i(S[\xi_1]-S[\xi_2])} e^{i\lambda \int_0^t \partial_{t'} Q(\xi_1(t'))\mathrm{d}t'}$$

If for example $Q = \theta(x)$, then:

$$\chi(\lambda) = \int_{\substack{\xi_1(0)=\xi_2(0) \\ \xi_2(T)=\xi_1(T)}} \mathcal{D}[\xi_1]\mathcal{D}[\xi_2]\rho(\xi_1(0), \xi_2(0)) e^{i(S[\xi_1]-S[\xi_2])} e^{i\lambda \int_0^t \int_0^T \mathrm{d}x \partial_{t'} |\xi_1(t', x)|^2 \mathrm{d}t'}$$

Substitution of $i\partial_{t'}\xi_1(t') = H\xi_1(t')$, we get $\int I(t')\mathrm{d}(t')$ instead of $\int_0^t \partial_{t'}|\xi_1(t')|^2\mathrm{d}t'$
$\Rightarrow$ By definition of the path integral will get a time ordered exponent of current operators.
However: substitution is legitimate only for classical trajectories in the path integral, thus describes only the saddle point of the integral.

Moments

- We are interested in the cummulants defined by

$$<< Q^k >>= i^k \partial_\lambda^k \log \chi(\lambda)|_{\lambda=0}$$

- Representation of the differentiations:

  Consider words over Z_2, with cyclic permutations identified, and the operator D defined by the rules:
  A) $D(1) = -(11) + (0)$
  B) $D(0) = -(10) - (1)$
  C) D satisfies the Leibniz rule: $D(ab) = (Da)b + a(Db)$,

$$\begin{aligned} D(1) &= -(11) + (0) \\ D^2(1) &= 2(111) - 3(10) - (1) \\ D^3(1) &= 6(1111) - 12(110) - 3(00) + (11) - (0) \end{aligned} \tag{14}$$

  Then the $(k+1)$-th cummulant is related to $D^k(1)$:
  Replace $1 \to n(\hat{Q}_T - \hat{Q})$ and $0 \to n((\hat{Q}_T - \hat{Q})^2 + [\hat{Q}_T, \hat{Q}])$, and trace the resultant operator.

Transport: First moment

- $D^0(1) = (1) \Rightarrow$

$$<< Q >>= -i\text{Tr}(n_d(\widehat{Q}_T - \widehat{Q})) = -i\text{Tr}((U^\dagger n_d U - n_d)\widehat{Q}) \tag{15}$$

- In the adiabatic limit:

$$<< Q >>= -i\text{Tr}((S^\dagger n_d S - n_d)\widehat{Q}) \tag{16}$$

Now we use that

$$i\hbar \dot{S}_d = [H_0, S_d] \tag{17}$$

So that

$$S_d H_0 S_d^\dagger = H_0 - \mathcal{E} \tag{18}$$

- Where $\mathcal{E} = i\hbar \dot{S}_d S_d^\dagger$ is called the energy shift.
A conjugate notion to Wigner time delay $\mathcal{T} = i\hbar(\partial_E S_d) S_d^\dagger$

- It follows that

$$S_d n(H_0) S_d^\dagger = n(H_0 - \mathcal{E}) \tag{19}$$

- In the limit of adiabatic variation of the scattering we have

$$\mathcal{E} = i\hbar \dot{S}_d S_d^\dagger << 1$$

and

$$< Q >= \mathrm{Tr}(n(H_0 - \mathcal{E}) - n(H_0))\hat{Q} \simeq \mathrm{Tr}(n'(H_0)\mathcal{E}\hat{Q}) = q \int \mathrm{d}t \int \mathrm{d}E n'(E) \mathcal{E}_{11}(t)$$

- Note $n'(H_0)$ is localized at the fermi energy.

- Equivalent to the result of
M. Büttiker, A. Prêtre and H. Thomas, Phys. Rev. Lett. **70**, 4114 (1993).

Noise: Second moment

- Noise is the variance per unit time of the transfer distribution :

$$< (\Delta Q)^2 >= - << Q^2 >>= \mathrm{Tr}\big(n(\hat{Q}_T - \hat{Q})\big(1-n\big)(\hat{Q}_T - \hat{Q})\big), =$$
$$\mathrm{Tr}\big(n(1-n)\,(\hat{Q}_T - \hat{Q})^2\big) + \tfrac{1}{2}\,\mathrm{Tr}\big([n,(\hat{Q}_T - \hat{Q})]\,[(\hat{Q}_T - \hat{Q}),n]\big).$$

It splits into two positive terms:

- **Johnson-Nyquist** noise is the first term, proportional to temperature:

$$Q^2_{JN} = \mathrm{Tr}\big(n(1-n)\,(\hat{Q}_T - \hat{Q})^2\big) = -T\,\mathrm{Tr}(n'(\hat{Q}_T - \hat{Q})^2) \geq 0, \qquad (20)$$

- The **quantum shot noise** involves correlations at different times and survives at $T = 0$ is the second term:

$$Q^2_{QS} = \frac{1}{2}\,\mathrm{Tr}\big([n,\hat{Q}_T]\,[\hat{Q}(T),n]\big) = \frac{1}{2}\,\mathrm{Tr}\big([\delta n,Q]\,[Q,\delta n]\big) \geq 0 \qquad (21)$$

Classical limit of the commutator is order $\hbar \Rightarrow Q^2_{QS} \to 0$ in this limit.

Noise: third moment

- Importance of the third moment:
  L. S. Levitov and M. Reznikov, cond-mat/0111057.

- The third cummulant is obtained from $D^2(1) = 2(111) - 3(10) - (1)$

$$<< Q^3 >>= -i\text{Tr}(-2n_d(\hat{Q}_T - \hat{Q})n_d(\hat{Q}_T - \hat{Q})n_d(\hat{Q}_T - \hat{Q}) + 3n_d(\hat{Q}_T - \hat{Q})n_d(\hat{Q}_T - \hat{Q})^2 - n_d(\hat{Q}_T - \hat{Q})) \tag{22}$$

- Odd moments always have a term proportional to the first moment.

- **Motivation to study the Fourth moment:** Until now all of the moments didn't contain explicitly the term $[\hat{Q}_T, \hat{Q}]$.

  A check reveals that $<< Q^4 >>$ does contain this term.

  This term measures an "uncertainty" between measuring a particles side and the knowledge of where it originated.

The many cycle limit:
When is the pumping "extensive" in time?

- Notion of extensivity - all moments?
- For periodic driven systems we denote $\Lambda_m = U^{m\dagger} e^{i\lambda Q} U^m e^{-i\lambda Q}$ where U is a one cycle evolution, and denote $\chi_m = \det(1 + n(\Lambda_m - 1))$.
- Quantities averaged over many cycles are computed from $\frac{1}{m} \log \chi_m$.
- The equation for extensivity is $\chi_{m+l} \sim \chi_m \chi_l$:

$$\det(1 + n(\Lambda_{l+m} - 1)) \sim \det(1 + n(\Lambda_l - 1)) \det(1 + n(\Lambda_m - 1)) \quad (23)$$

- This equation doesn't imply an equation for the operators. Let's guess:

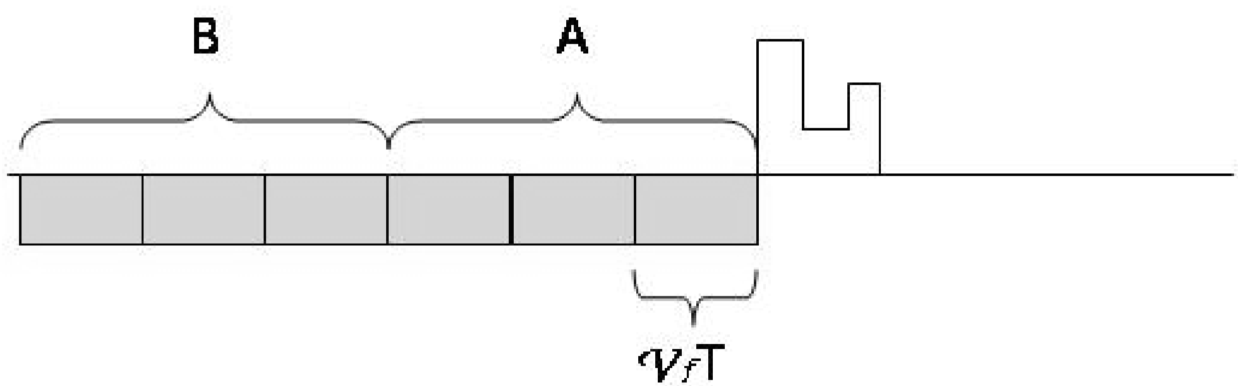
$$1 + n(\Lambda_{l+m} - 1) \sim U^{m\dagger}(1 + n(\Lambda_l - 1)) U^m (1 + n(\Lambda_m - 1)) \quad (24)$$

- Extensivity in time is a property of steady state pumping.
  Under the condition: $[n, U] = 0$, extensivity is equivalent to:

$$\underbrace{U^m(\Lambda_l - 1)U^{m\dagger}}_{B}\underbrace{(\Lambda_m - 1))}_{A}\,n(n-1) \tag{25}$$

- $n(n-1)$ is a function localized at the Fermi energy $\Rightarrow$ contribution only from states travelling approximately at V_F.

  A is non-vanishing on states that reach the pump during m cycles.
  B is non vanishing on states that reach the pump between cycles $m, l+m$.



The overlap is a "boundary" term.

Further projects:

Meaning of the fourth moment.

Interactions

Non adiabatic problems (microwave radiation)

Statistics of Spin transport