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Quantum Chaos in Nanostructures

A.I. Larkin

Outline:

- **Introduction**

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  b) Ehrenfest time
  c) Weak localization
  d) Shot Noise

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  b) RMT result for DoS in Andreev Billiards
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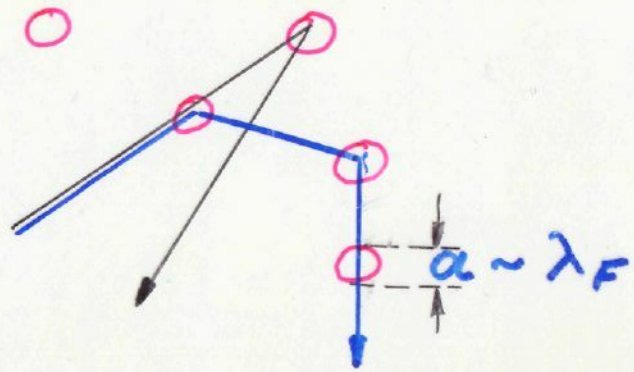
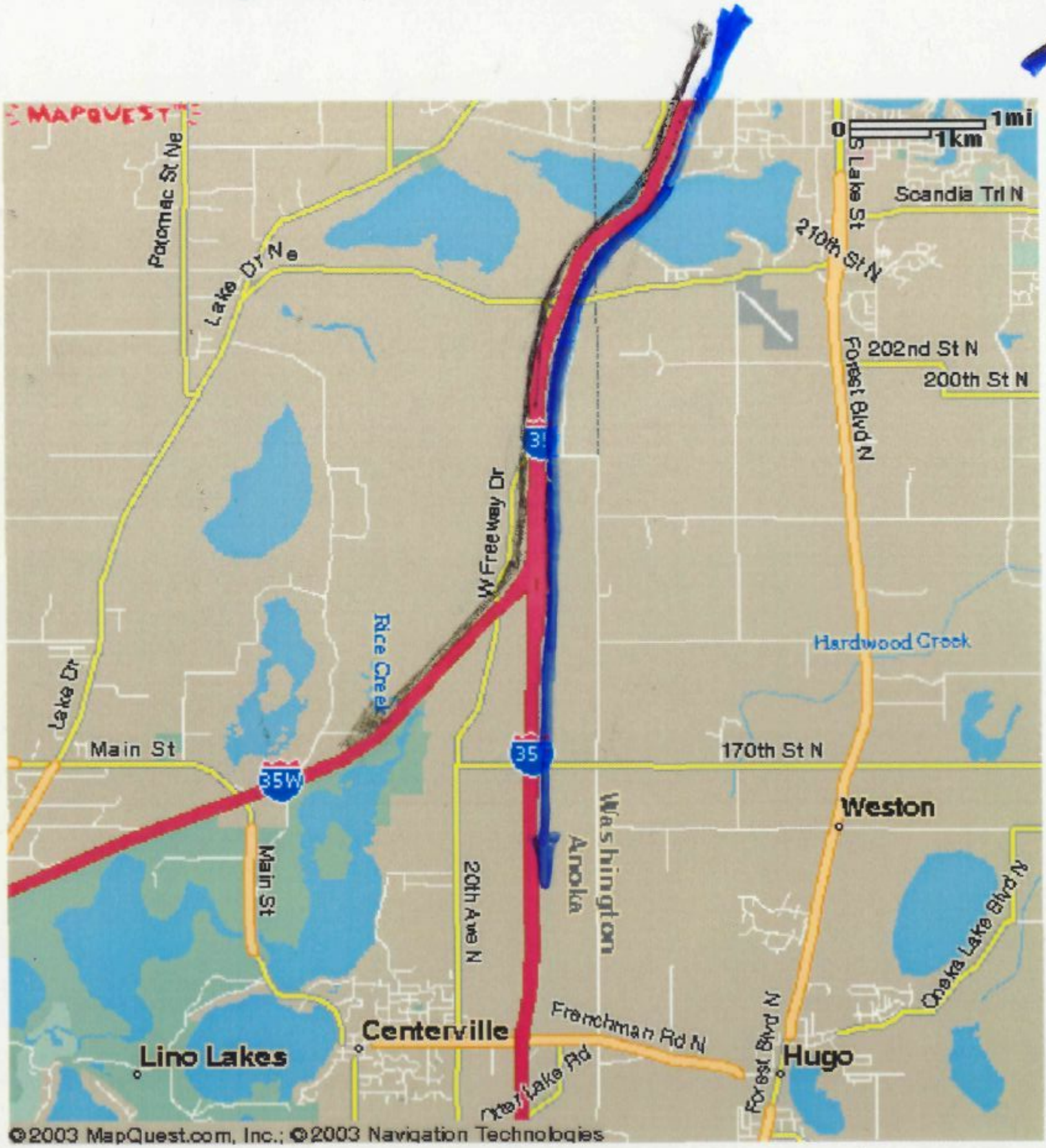
- **Energy Level Statistics**

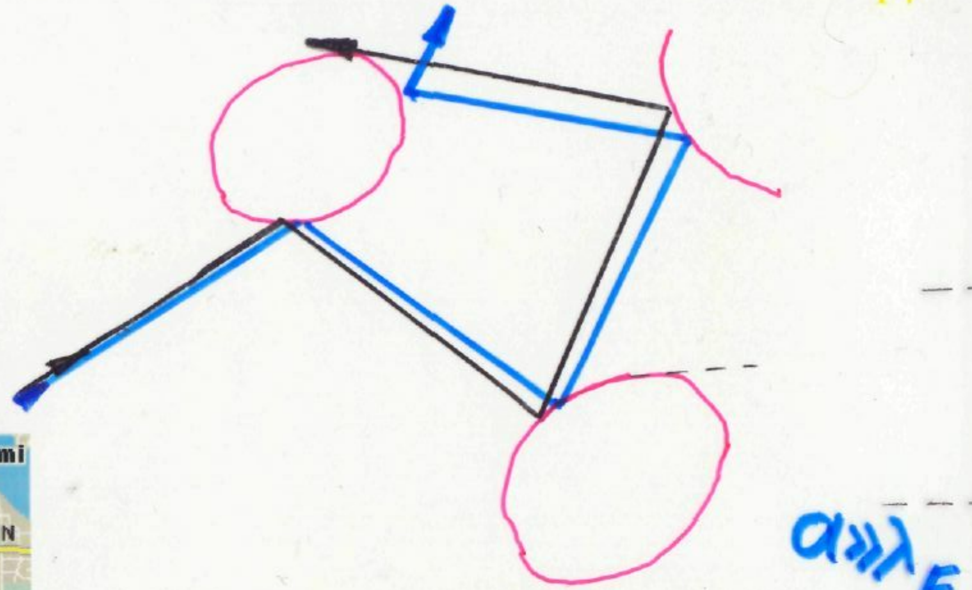
  a) RMT
  b) Non-linear Sigma model
  c) Periodic Orbits
  d) Crossover Regime

QUANTUM DISORDER

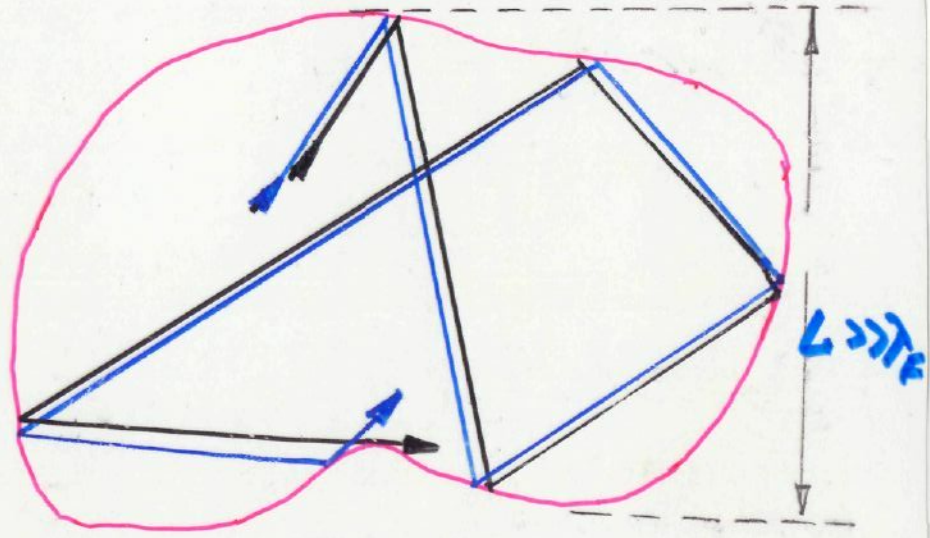


QUANTUM CHAOS





$a \gg \lambda_F$



$L \gg \lambda_F$

QUANTUM DOT

ERENFEST TIME

$$\frac{d\langle p\rangle}{dt} = -\langle \frac{dU}{dx} \rangle$$ Erenfest equation

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$$

t_E - time of life of wave pocket

In the semiclassical chaotic sistem

$$\langle [p_z(0), p_z(t)]^2 \rangle \sim \hbar^2 \langle \left(\frac{\partial p_z(t)}{\partial z(0)}\right)^2 \rangle \sim$$

$$\sim \hbar^2 e^{2\lambda t} \qquad t < t_E$$

λ - Lyapunov exponent

$$t_E = \frac{1}{\lambda} \ln 1/\hbar$$

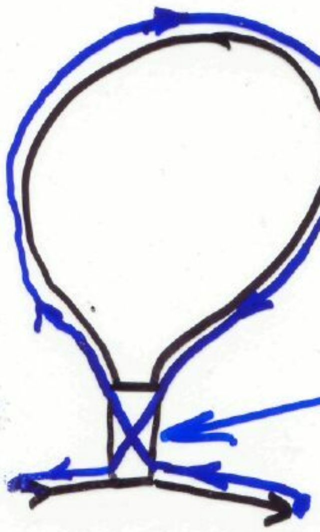
(Larkin, Ovchinnikov JETP 28, 1200 (1969))

for kicked rotator $t_E \sim \ln 1/\hbar$ also

(Berman, Zaslavsky Physica A 29, 2415 (1978))

Weak-localization

I. Aleiner A. Larkin (1996)



Cooperon for 2D

$C_0(t) = \frac{1}{Dt}$ $\quad C_0(\omega) = \frac{\ln \frac{1}{\omega\tau}}{D}$

Hikami box

$\Gamma(t) = \delta(t - t_E)$

$\Gamma(\omega) = e^{i\omega t_E}$

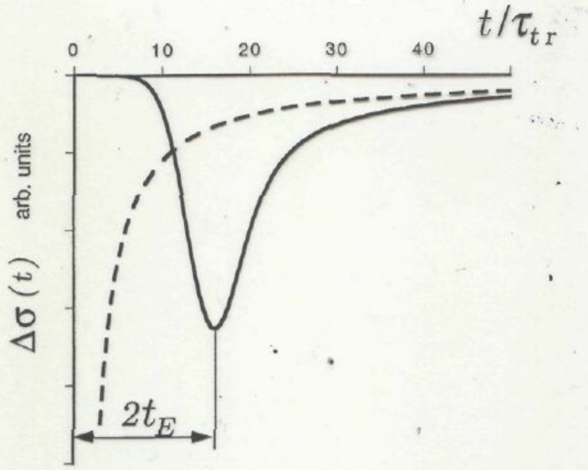


FIG. 1. The weak-localization correction to the conductivity in the time domain, $\Delta\sigma(t)=\int(d\omega/2\pi)\Delta\sigma(\omega)e^{-i\omega t}$ for the quantum chaos (solid line) and quantum disorder (dashed line) regimes. The developed theory is valid for $t\gtrsim t_E$. Solid curve is calculated for parameters $\lambda=4\lambda_2=1/\tau_{tr}$ and $\ln(a/\lambda_F)=7$.

$$\Gamma(\omega)=\exp\left(2i\omega t_E-\frac{2\omega^2\lambda_2 t_E}{\lambda^2}\right), \tag{1.2}$$

where the Ehrenfest time t_E is the time it takes for the minimal wave packet to spread over the distance of the order of a, and is given by[1]

$$t_E=\frac{1}{\lambda}\ln\left(\frac{a}{\lambda_F}\right). \tag{1.3}$$

Quantity $\lambda_2\simeq\lambda$ in Eq. (1.2) characterizes the deviation of the Lyapunov exponents, and it will be explained in Sec. II in more detail. In the time representation, result (1.2) corresponds to the delay of the weak-localization correction to the current response by large time $2t_E$; see Fig. 1.

The paper is arranged as follows. In Sec. II, we present the phenomenological derivation of Eq. (1.2). The explicit expression relating the weak-localization correction to the solution of the Liouville equation will be derived in Sec. III. In Sec. IV, we will find the quantum corrections to the conductivity in the infinite chaotic system. Section V describes the effects of the magnetic field and finite phase relaxation time on the renormalization function. The conductance of the ballistic cavities is studied in Sec. VI. Our findings are summarized in Sec. VIII.

II. QUALITATIVE DISCUSSION

The classical diffusion equation is based on the assumption that at long time scales an electron loses any memory about its previous experience. However, during its travel, the electron may traverse the same spatial region and be affected by the same scatterer more than once. These two scattering events are usually considered independently, because with the dominant probability the electron enters this region having completely different momentum.

However, if we wish to find the probability $W_0(T,\rho_0)$ for a particle to have a momentum opposite to the initial one, $\mathbf{p}(T)=-\mathbf{p}(0)$ (time T is much larger than τ_{tr}), and to

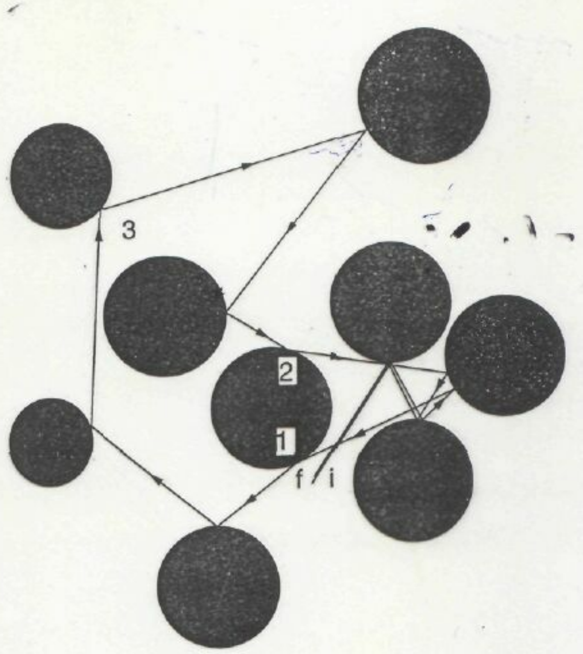


FIG. 2. The classical trajectory corresponding to the probability of return at the initial point with the momentum opposite to the initial one. In the "Lyapunov region" the initial "i-1" and final "2-f" fragments of the trajectory are governed by the same potential.

approach its starting point at small distance $|\mathbf{r}(T)-\mathbf{r}(0)|=\rho_0\ll a$, we should take into account the fact that the motion of the particle at the initial and final stages are correlated. This is because the trajectory along which the particle moves on the final stage $[\mathbf{r}(T-t),\mathbf{p}(T-t)]$ almost coincides with the trajectory particle moving along at the initial stage $[\mathbf{r}(t),\mathbf{p}(t)]$; see Fig. 2. These correlations break down the description of this problem by the diffusion equation. The behavior of the distribution function for this case can be related to the Lyapunov exponent, and we now turn to a discussion of such a relation. [The relevance of $W_0(T,\rho_0)$ to the weak-localization correction will become clear shortly.]

The correlation of the motion at the final and initial stages can be conveniently characterized by two functions

$$\boldsymbol{\rho}(t)=\mathbf{r}(t)-\mathbf{r}(T-t),\quad \mathbf{k}(t)=\mathbf{p}(T-t)+\mathbf{p}(t). \tag{2.1}$$

The classical equations of motion for these functions are

$$\frac{\partial\boldsymbol{\rho}}{\partial t}=\frac{\mathbf{k}(t)}{m}, \tag{2.2a}$$

$$\frac{\partial\mathbf{k}}{\partial t}=\frac{\partial U[\mathbf{r}(T-t)]}{\partial\mathbf{r}}-\frac{\partial U[\mathbf{r}(t)]}{\partial\mathbf{r}}, \tag{2.2b}$$

where U is the potential energy. If the distance ρ is much larger than the characteristic spatial scale of the potential a, Eqs. (2.2) lead to the usual result $\langle\rho(t)\rangle\propto t^{1/2}$ at times t much larger than τ_{tr}. The situation is different, however, for $\rho\ll a$, where the diffusion equation is not applicable (we will call this region of the phase space the "Lyapunov region"). Thus the calculation of function $W_0(T,\rho_0)$ should be performed in two steps. First, we have to calculate the conditional probability $W(a,\rho_0;t)$, which is defined so that the

$$(2i\omega + \mathcal{L}_1 + \mathcal{L}_2)\, C(1,2) = 0$$

$$\mathcal{L} = p\frac{\partial}{\partial R} - \nabla U \frac{\partial}{\partial p}$$ Liouville operator

$$C(1;2) = C(R+\rho, p+\varphi;\ R-\rho, -p+\varphi)$$

$$z = \ln\left(\varphi^2 + \left(\frac{\rho}{a}\right)^2\right)^{1/2}$$

for $\varphi \sim 1,\ \rho \sim a;\ z \sim 1$ $\quad C = C_0 = \int \frac{d^2q}{2i\omega + Dq^2} = \frac{\ln\omega}{D}$

for $\varphi \ll 1$, $\rho \ll a$, $-z \gg 1$

$$\left(2i\omega + \lambda\frac{\partial}{\partial z} + \frac{\lambda_2}{2}\frac{\partial^2}{\partial z^2} + \frac{e^{-2z}}{2\tau_1}\gamma\frac{\partial}{\partial z}\right) C(z) = 0$$

$$\langle C(1,\bar{1}) = \lim_{z \to -\infty} C(z) = \Gamma(\omega)\, C_0(\omega)$$

SHOT NOISE

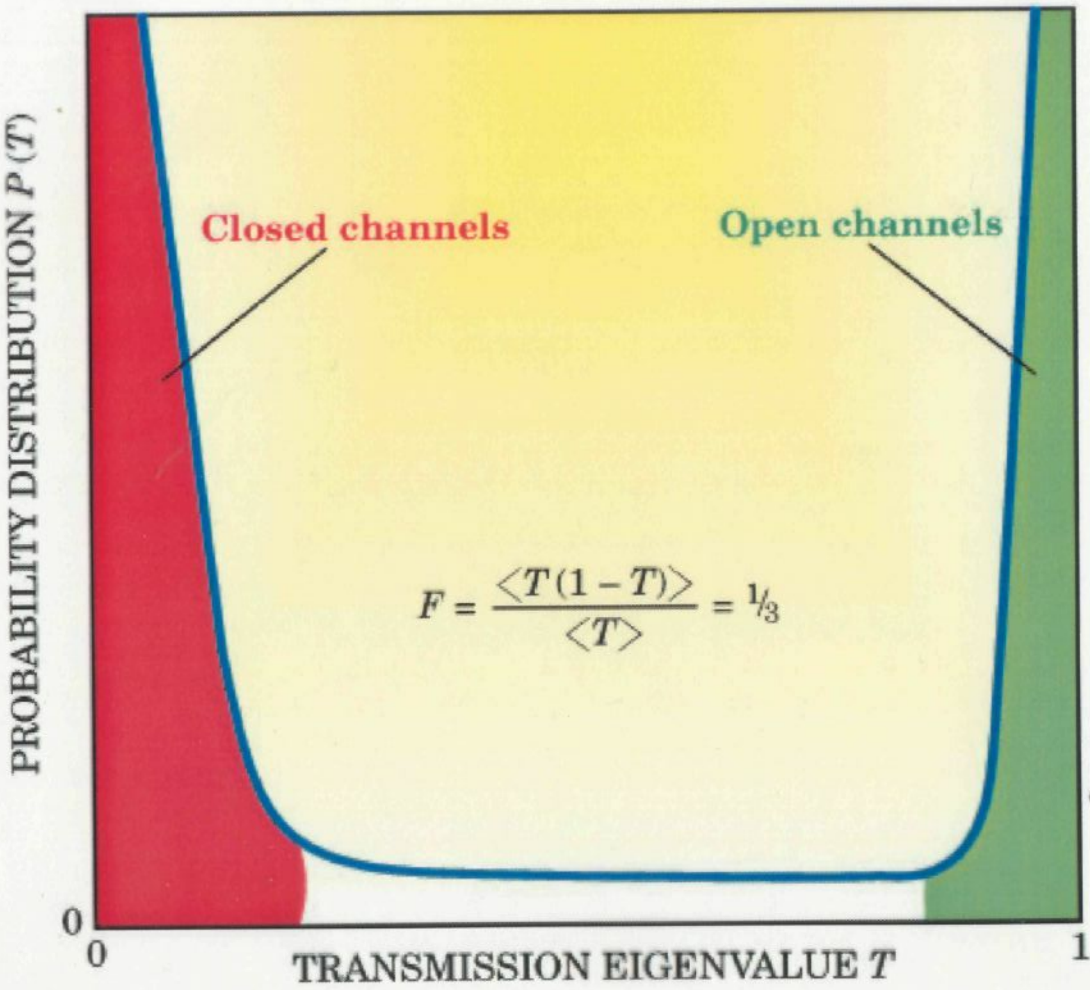
$S =$

$= \int \left(\langle I(t) I(0) \rangle - \langle I \rangle^2 \right) dt =$

$= 2e \frac{2e^2}{\hbar} V \sum_{n=1}^{N} T_n (1 - T_n)$

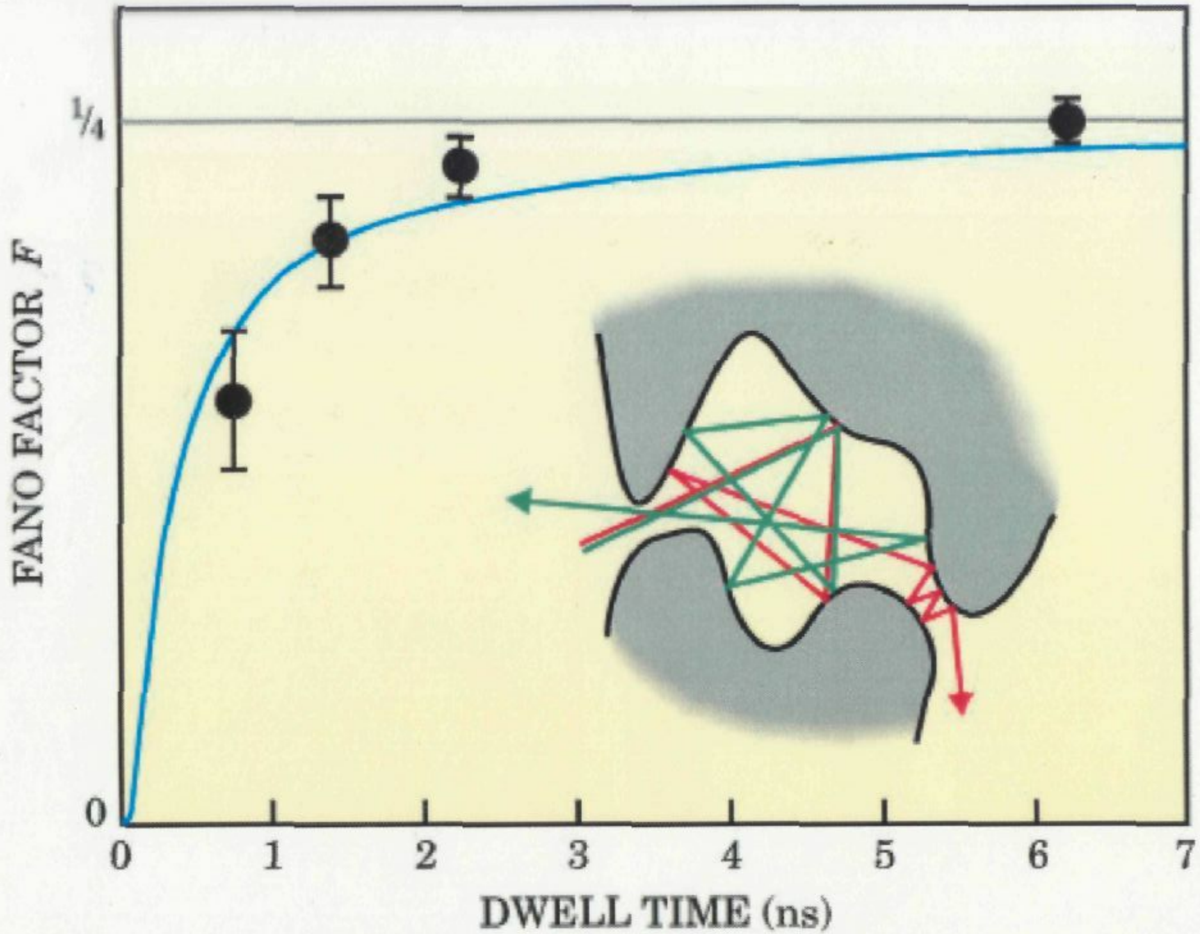
Khlus JETP, 66, 1243 (1987)

Lesovik JETP Lett, 49, 592 (1989)



$S = 2e \langle I \rangle \cdot F$

The functional form of the distribution (derived by Oleg Dorokhov) is $P(T) \sim T^{-1}(1 - T)^{-1/2}$. The Fano factor follows directly from the ratio $\int T^2 P(T) dT / \int T P(T) dT$, which takes on a universal value.



Dependence of the Fano factor F of an electron billiard on the average time τ_{dwell} that an electron dwells inside the cavity. The data points were measured in a two-dimensional electron gas confined to an irregularly shaped region *[Oberholzer et al., Nature 2002]*, and the solid curve is the theoretical prediction $F = \frac{1}{4}\exp(-\tau_E/\tau_{dwell})$ [*Agam et al. PRL 2000*] for the transition from stochastic to deterministic scattering, with Ehrenfest time $\tau_E = 0.27$ ns as a fit parameter.

Tiny variations in the electron's path (red or green) determine where it exits.

Quantum Disorder and Quatum Chaos in Andreev Billiard

Vavilov, Larkin Phys. Rev. B 67, 115335 (2003).

1. Method of classical tragectory in the theory of superconductivity

$$\langle j \rangle = \frac{e}{m} p_{nm} \hat{G}_m (pA)_{ml} \hat{\sum_l} (\delta_{ln} + (Ap)_{lk} \hat{G}_k (Ap)_{kn} \hat{G}_n^{+}) =$$

$$= \frac{e}{m} \sum_{\text{tragectory}} \int p(t_1) \hat{G}_{t_1 t_2} (p_{t_2} A) \hat{G}_{t_2 t_3} [\delta_{t_3 t_1} +$$

$$+ (Ap_3) \hat{G}_{t_3 t_4} (Ap(t_4)) \hat{G}_{t_4 t_1} + \dots]$$

De Gennes, Tinkham Physics 1. 107 (1964)
Shapoval JETP 20, 675 (1965) 22, 647 (1966)

$$\langle p(0) p(t) \rangle \sim e^{-t/\tau_{tr}}$$

but $\langle p_0 p_t p_0 p_t \rangle \neq \langle p_0 p_0 p_t p_t \rangle$ for $t > t_E$

(A.L. Ovchinnikov 1969)

$$\hat{G}_{tt} = \int G \, d\xi = \begin{pmatrix} g & f_+ \\ f_- & -g \end{pmatrix} =$$

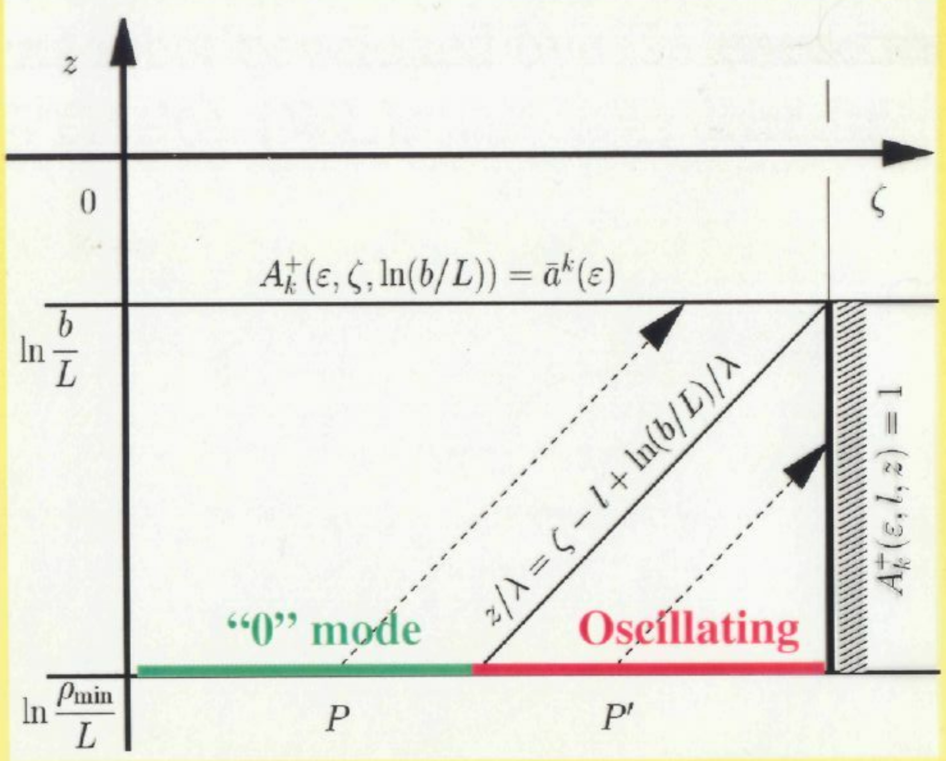
$$= \frac{1}{1 + a_+ a_-} \begin{pmatrix} 1 - a_+ a_- & 2a_+ \\ 2a_- & a_+ a_- - 1 \end{pmatrix}$$

$$\hat{G}^2 = 1$$

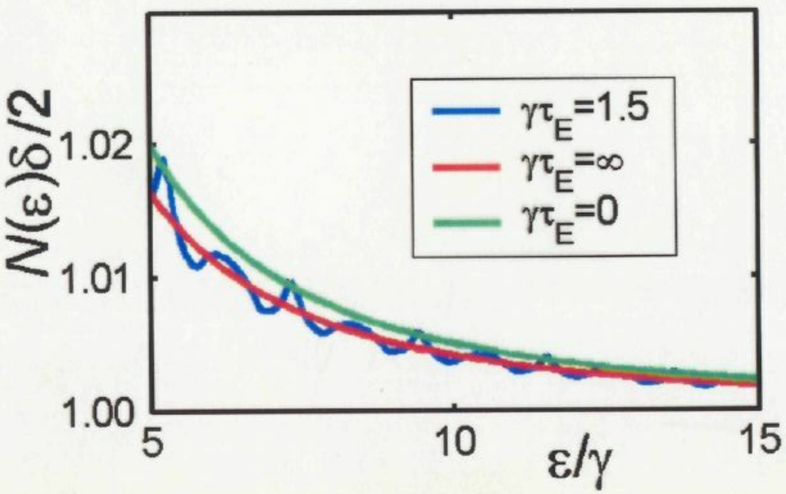
Wave packet evolution

The boundary conditions and the characteristics of the equation for $A^+_k(\varepsilon,\zeta,z)$:

$$\left(2i\varepsilon k - \frac{\partial}{\partial\zeta} - \lambda\frac{\partial}{\partial z}\right) A_k^+(\varepsilon,\zeta,z) = 0$$



The DoS has oscillating components with the period $\propto\tau_E^{-1}$.



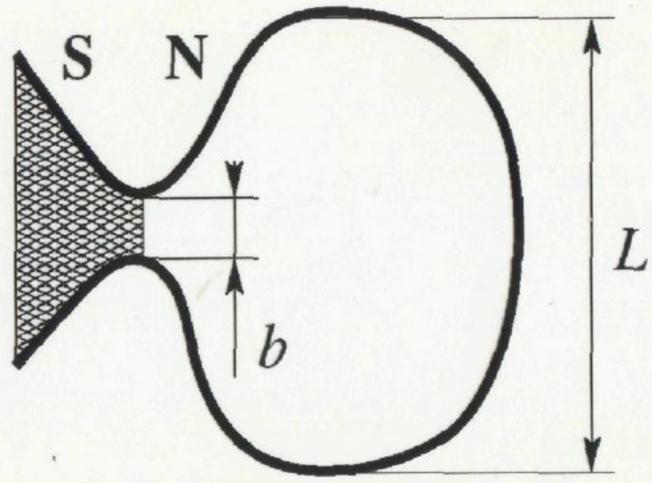
The density of states at $\varepsilon \gg \gamma$ again may be represented as

$$\mathcal{N}(\varepsilon) = \frac{2}{\delta}\left(1 + \phi(\varepsilon\tau_0)\frac{\gamma^2}{2\varepsilon^2}\right),$$

where $\phi(\varepsilon\tau_0)$ is

$$\phi(\varepsilon\tau_0) = 1 - \sum_{k=2}^{\infty}\frac{(-1)^k}{k^2}\mathrm{Re}\left\{1 - \exp\left(-\left(2ik\varepsilon + \gamma + \frac{2k^2\lambda_2\varepsilon^2}{\lambda^2}\right)\tau_E\right)\right\}^2$$

Andreev billiards: Model



The dot is characterized by the size L of the normal region, the size of the SN contact $b \ll L$.

Electrons in the normal region have the Fermi length λ_F and mean free path l.

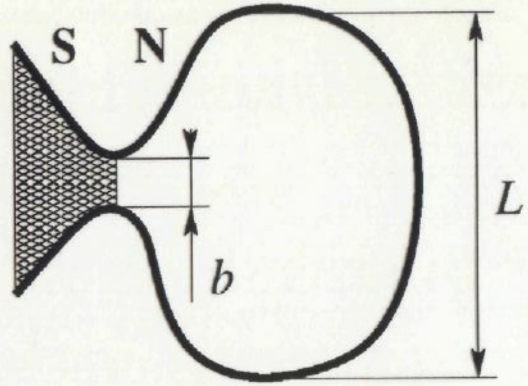
An Andreev billiard is a small metal grain or a semiconductor quantum dot connected to a superconductor.

At the SN boundary electrons are reflected as holes according to the Andreev mechanism.

What is the electron density of states in the normal region?

Parameters of an Andreev Billiard

Four length scales characterize a billiard:



- size L of the normal region;
- size b of the SN contact, $b \ll L$;
- electrons Fermi length λ_F;
- mean free path l.

Particularly, the length scales determine the energy scales:

- Fermi energy: $E_F \propto v_F/\lambda_F$;
- scattering rate (inverse mean free time): $1/\tau_0$;
- Thouless energy (inverse ergodic time): $E_{Th} = 1/\tau_{erg}$
  [dirty metal ($l \ll L$) $E_{Th} \propto v_F l/L^2$; clean metal ($l \gg L$) $E_{Th} \propto v_F/L$];
- escape rate: $\gamma = (b/L)\, E_{Th} \ll E_{Th}$;
- mean level spacing $\delta \propto \hbar^2/mL^2$, where m is the electron mass.

Below we always assume that $\delta \ll \gamma \ll E_{Th}$.

Semiclassical approach

[Melsen et al., Europhys. Lett.; Schomerus and Beenakker, PRL ***82****, 2951]*

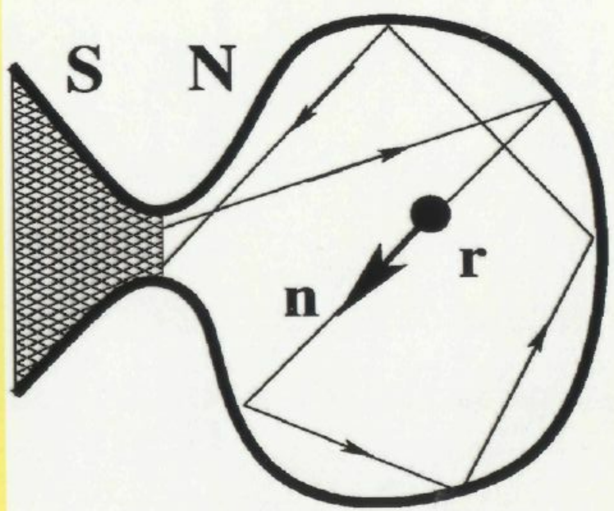
An electron and a hole form a standing wave along a trajectory of length s with its terminals at the SN contact. Their energy is

$$E_n = (n + 1/2)\frac{\pi\hbar v_F}{s}.$$

DoS is given by averaging of spectrum E_n with respect to the distribution $P(s)=\alpha\exp(-\gamma s/v_F)$:

$$\mathcal{N}(\varepsilon) \propto \int_0^\infty ds P(s) \sum_{n=0}^{\infty} \delta(\varepsilon - E_n).$$

The result is $\mathcal{N}(\varepsilon) = \frac{2}{\delta_1}\frac{\pi^2\gamma^2}{4\varepsilon^2}\frac{\cosh \pi\gamma/2\varepsilon}{\sinh^2 \pi\gamma/2\varepsilon}$



(**r**,**n**) is determined by position **r** and direction **n**.

The Eilenberger equation reproduces the RMT result:

The scattering term depends on the average of the Green function over the phase space. The Eilenberger equation is lir

$$\mathcal{L}\hat{\mathcal{G}}(\varepsilon,\mathbf{r},\mathbf{n}) = \left[\left(i\varepsilon\hat{\tau}_z + \frac{1}{2\tau_0}\langle\hat{\mathcal{G}}(\varepsilon,\mathbf{r},\mathbf{n})\rangle\right), \hat{\mathcal{G}}(\varepsilon,\mathbf{r},\mathbf{n})\right]$$

Solving the linear equation, we obtain the self consistency equation, which gives:

- The semiclassical result at $\gamma\tau_0 \gg 1$;
- The RMT result at $\gamma\tau_0 \ll 1$.

This equation also allows us to consider the classical-to-quantum crossover.

[M.Vavilov and A.I. Larkin, PRB 2003]

Gap fluctuations

RMT predicts a square root singularity near the excitation gap:

$$\rho_{\mathrm{mf}}(\varepsilon) \propto \rho_0 \sqrt{\frac{\varepsilon}{E_{\mathrm{g}}} - 1}.$$

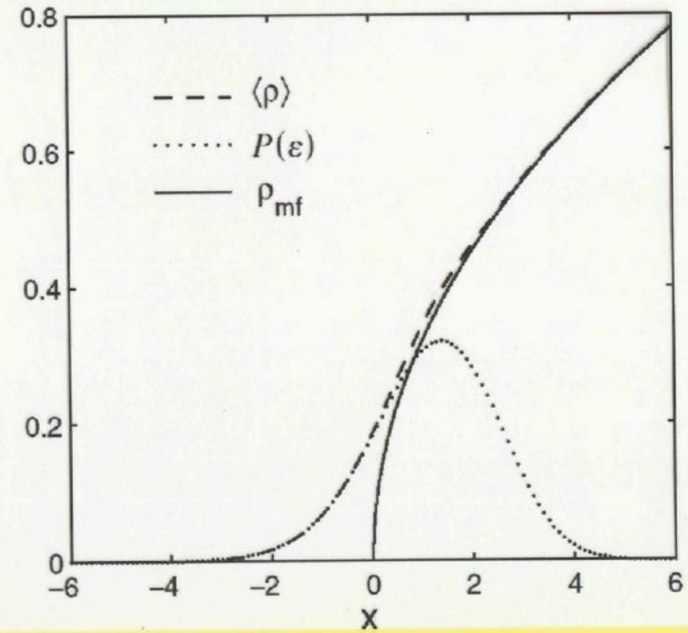
This result does not take into account discreteness of the energy levels.

Fluctuations of single energy levels smear the singularity.

Energy levels distribution is universal and described by the energy scale $\Delta=\gamma^{1/3}\delta^{2/3}$.

(M.Vavilov *et al,* PRL 86, 876;

P.M.Ostrovsky *et al,* PRL 87, 027002.)



The figure shows the actual ensemble average (dashed) and mean field (solid) DoS as a function of $x=(\varepsilon-E_g)/\Delta$. The dotted line represents the distribution of the lowest energy level.

Chaotic Andreev billiards – effect of quantum diffraction

In systems with Quantum Disorder the mixing in the phase space occurs after a single scattering off an impurity.

In clean chaotic systems mixing in the phase space occurs within a small region due to the quantum diffraction.

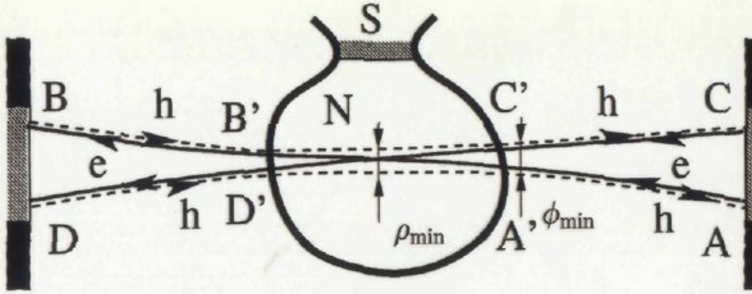
The Ehrenfest time τ_E plays the role of the mean free time τ_0.

$$\tau_{\mathrm{E}} \propto \frac{1}{\lambda} \ln \frac{L}{\lambda_{\mathrm{F}}},$$

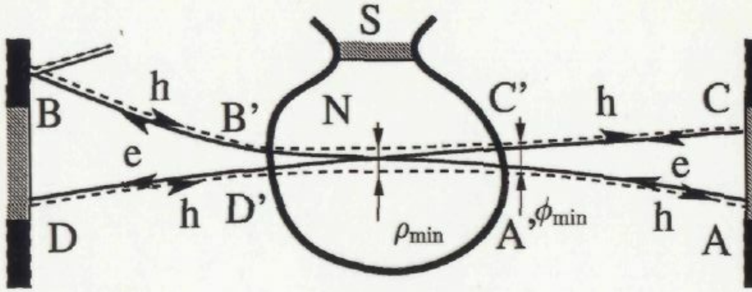
where λ is the Lyapunov exponent, L is the system size, and λ_F is the Fermi wavelength: $\lambda_F \propto \hbar$.

The Ehrenfest time depends logarithmically on the ratio of the quantum scale and the classical scale L.

In clean systems the Quantum diffraction is important!



a)



b)

Due to the uncertainty principle, the minimal wave packet is

$$\phi_{\min} = \sqrt{\lambda_F / L}, \qquad \rho_{\min} = \sqrt{\lambda_F L}$$

The distance between two trajectories grows in time with the Lyapunov exponent λ

$$\rho(t) = \rho_{\min} e^{\lambda t}, \qquad \phi(t) = \phi_{\min} e^{\lambda t}$$

The wave packet spreads over the whole phase space after t_E

$$t_E = \frac{1}{\lambda} \ln \frac{L}{\lambda_F} = \frac{2}{\lambda} \ln \frac{L}{\rho_{\min}}.$$

The relevant time scale is τ_E

$$\tau_E = \frac{1}{\lambda} \ln \frac{b}{\rho_{\min}} = \frac{t_E}{2} - \frac{1}{\lambda} \ln \frac{L}{b}.$$

Size of the wave packet is equal to the contact size.

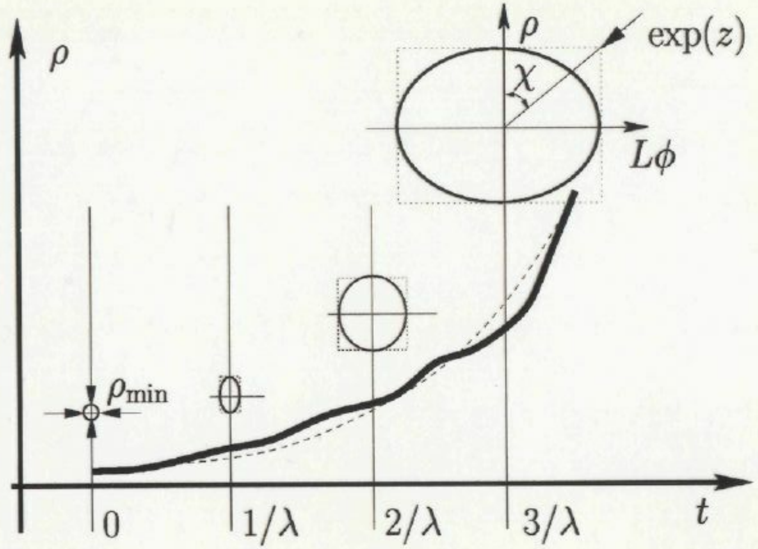
The procedure of calculations is similar to that developed by I.L. Aleiner and A.I. Larkin, PRB 14423, 54 (1996) [WL].

We introduce function

$$A_k^{\pm}(\varepsilon,\{\mathbf{x}_i\}) = \prod_{i=1}^{k} a_{\pm}(\varepsilon,\mathbf{x}_i)$$

For k=2 we have

$$z = \frac{1}{2}\ln\frac{\phi^2L^2+\rho^2}{L^2}, \qquad \chi = \arctan\frac{\phi L}{\rho}.$$



Only z represents the size of the wave packet, all other variables are not important (average out for the ergodic time). We have:

$$\left(2i\varepsilon k - \frac{\partial}{\partial\zeta} - \lambda\frac{\partial}{\partial z} - \frac{\lambda_2}{2}\frac{\partial^2}{\partial z^2}\right)A_k^{+}(\varepsilon,\zeta,z) = 0.$$

The boundary condition $A_k^{\pm}(\varepsilon, z=\ln b/L) = \overline{a}_{\pm}^{k}(\varepsilon,\mathbf{x})$

The last term represents fluctuations of λ and is small at $\varepsilon \ll \sqrt{\lambda^2/\lambda_2\tau_{\mathrm{E}}}$,

Energy Gap

In the limit of large Ehrenfest time, $\tau_E\gamma \gg 1$,

$$E_g \approx \frac{\pi}{4\tau_E}\left(1 - \frac{1}{\tau_E\gamma}\right).$$

For small τ_E the gap decreases linearly in $\gamma\tau_E$

$$E_g \approx E_g^{rmt}(1 - 0.23\gamma\tau_E).$$

The DoS has a squire-root dependence

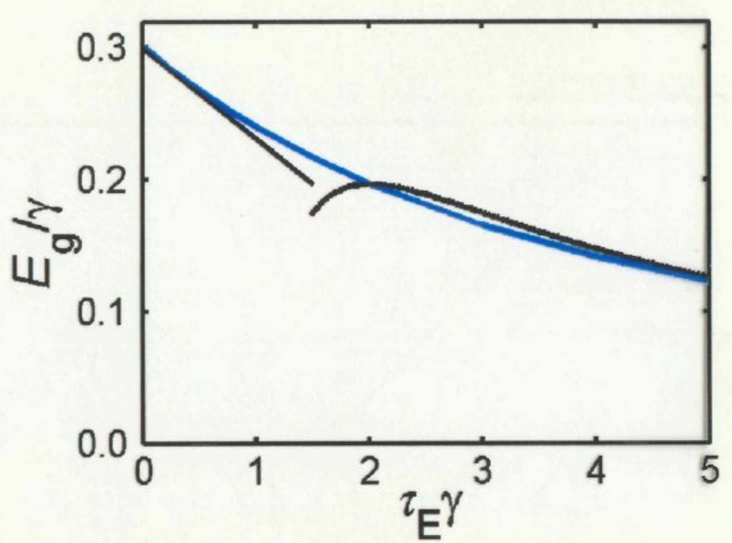
$$\mathcal{N}(\varepsilon) = \frac{c}{\delta_1}\sqrt{\frac{\varepsilon}{E_g} - 1}, \quad c = c_{rmt}(1 - 0.42\tau_E\gamma).$$

Jacquod et al. performed numerical test for a discrete phase space.

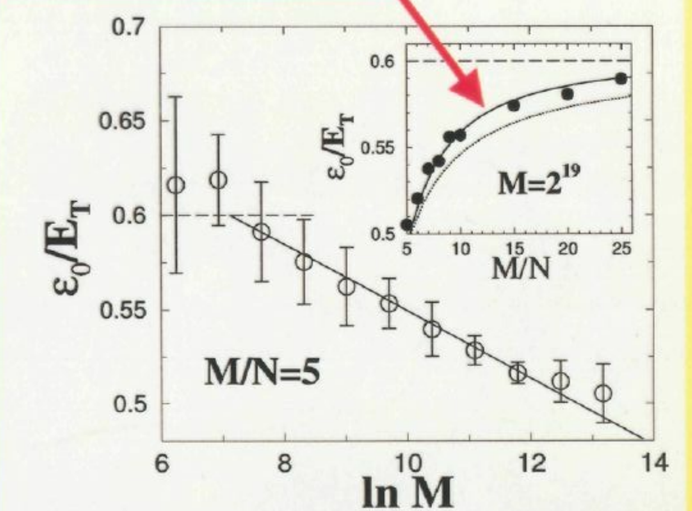
In their language, $L \propto$ M and $b \propto$ N.

Their result is consistent with

$$\tau_E = \frac{1}{\lambda}\ln\frac{b}{\sqrt{L\lambda_F}} \propto \ln N/\sqrt{M}$$



$\tau_E \propto \ln N/\sqrt{M}$



Jacuod, Schomerus, Beenakker, cond-mat/0212522.

Energy Level Statistics

a) Level Statistics in RMT (Dyson)

b) DoS in a metal grain (Gorkov, Eliashberg)

c) Supersymmetry (Efetov)

d) Periodic Orbits (Gutzwiller, Berry, Bogomolny, Keating)

e) Classical-to-Quantum Crossover

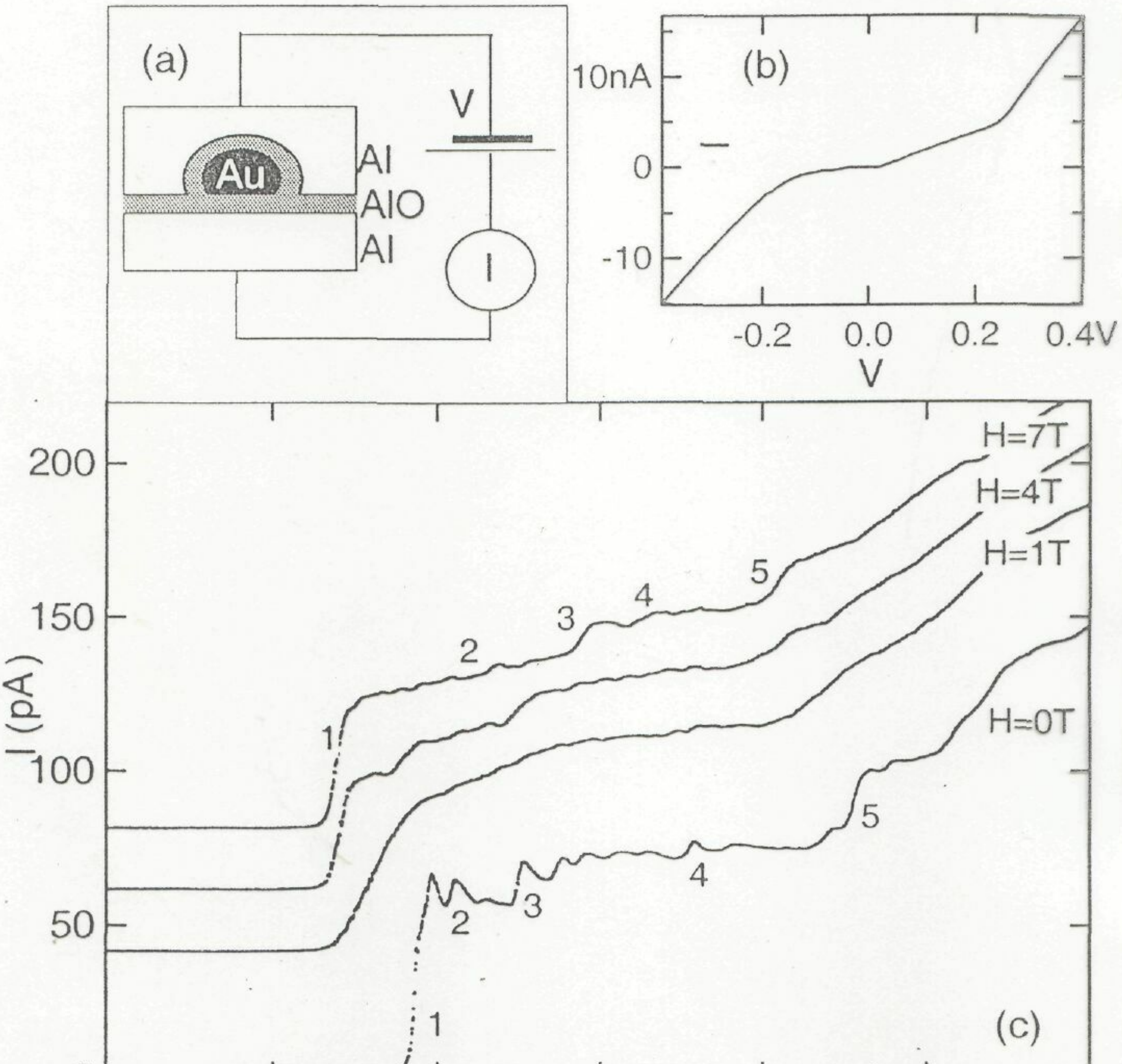
1) GOE (Aleiner, Larkin)

2) GUE (Tian, Larkin)

Fine Structure in Energy Spectra of Ultrasmall Au Nanoparticles

Dragomir Davidović* and M. Tinkham
Department of Physics and Division of Engineering and Applied Sciences
Harvard University, Cambridge, MA02138
(March 18, 2000)

We have measured the electronic energy spectra of Au nanoparticles of diameter 2-5 nm by tunnel spectroscopy, and found that the electron-in-a-box levels can be split into 2-10 energy levels, even in zero magnetic field. We propose that this splitting may result from spin multiplets in the presence of strong spin-orbit interaction. Relation to a theoretical model is discussed.



In the dot or grane

$$H = \frac{p^2}{2m} + U\left(\frac{x}{a}\right)$$

$$R = \frac{1}{64}\int \mathcal{D}Q \left(\int dx\, \mathrm{Str}\,\Lambda Q\right)^2 e^{-S}$$

$$S = \frac{\pi\nu}{2}\int dx\, \mathrm{Str}\left[\frac{i\omega}{2}\Lambda Q - T^{-1}\Lambda \mathcal{L} T + \frac{1}{\tau_q}\left(\frac{\partial Q}{\partial n}\right)^2\right]$$

Q. Disorder $a \sim \lambda_F$, $\frac{1}{\tau_q} > \omega$

$Q = \mathrm{const}$, $R = R_{Dyson}$ (Efetov)

Q. Chaos $a \gg \lambda_F$, $\frac{1}{\tau_q} \sim \frac{}{\hbar} \ll \omega$

Quantum levels in small metal particles.

Gorkov, Eliashberg 1965
JETP. 21. 940

Efetov 1980. JETP. 51. 1015

$$R(\omega)=\sum_{ij}\langle\delta(\varepsilon-\varepsilon_i)\,\delta(\varepsilon-\varepsilon_j-\omega)\rangle$$

Dyson Random Matrix Theory:

$$H=H_{mn} \qquad P(H_{nm})\sim e^{-H_{nm}^2}$$

Gaus ortogonal ensamble

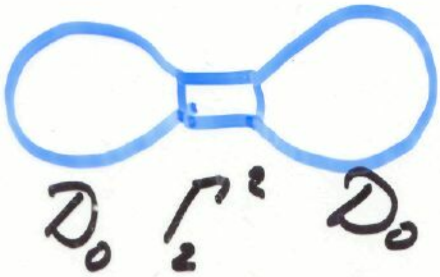
$$R^{o}(\omega)=1-\left(\frac{\Delta}{\pi\omega}\right)^2-\left(\frac{\Delta}{\pi\omega}\right)^4\left(1-\cos\frac{2\pi\omega}{\Delta}\right)-\ldots$$

Gaus unitary ensamble

$$R^{u}(\omega)=1-\left(\frac{\Delta}{\pi\omega}\right)^2\left(1-\cos\frac{2\pi\omega}{\Delta}\right)$$

GOE

$$\Delta R_p^0(\omega) = -\left(\frac{\Delta}{\pi}\right)^2 \frac{\partial^2 F_2(\omega)}{\partial \omega^2}$$

$F = i\omega$ 

$$\mathcal{D}_0 = \frac{1}{-i\omega}$$

$$R^0 = 1 - \frac{\Delta^2}{\pi^2\omega^2} - \frac{\Delta^3}{2\pi^2} \frac{\partial^2}{\partial\omega^2}\left(\frac{\sin 2\omega t_E}{\omega}\right) -$$

$$- \frac{\Delta^4}{2\pi^4\omega^4} \cos\frac{2\pi\omega}{\Delta}$$

GUE

$\Delta F =$  $+$ 

$$R^{U}(\omega) = 1 - \frac{\Delta^2}{\pi^2\omega^2}\sin^2\frac{\pi\omega}{\Delta} +$$

$$+\, 8\,\frac{\Delta^3}{\pi^3\omega^3}\,\sin\frac{2\pi\omega}{\Delta}\left(\cos 2\omega t_E - 1\right) +$$

$$+\,\frac{\Delta^4}{8\pi^4}\,\frac{\partial^2}{\partial\omega^2}\left(\frac{\cos 3\omega t_E - \cos 4\omega t_E}{\omega^2}\right)$$

Conclusion

In chaotic quantum dot, there appear a new scale, the Eherenfest time and new types of oscillations at $\omega \sim t_E^{-1} \gg \Delta$

The amplitudes of these oscillations are small.