Random Matrix Theory for the proximity effect in disordered wires

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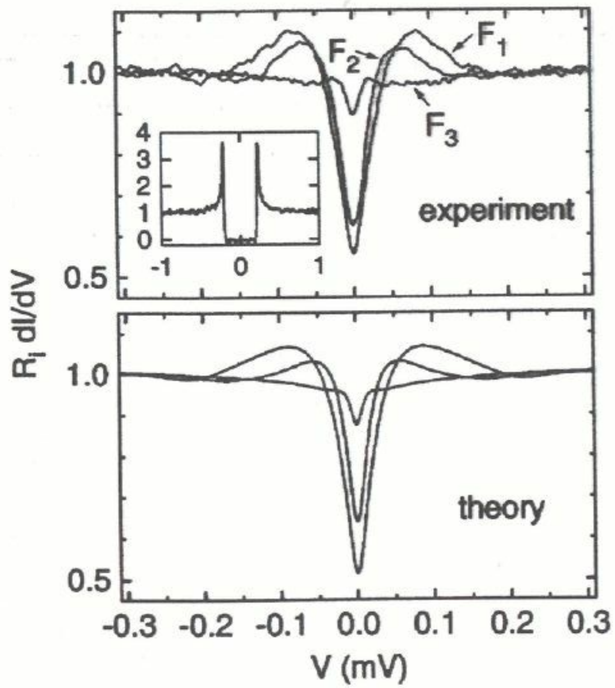
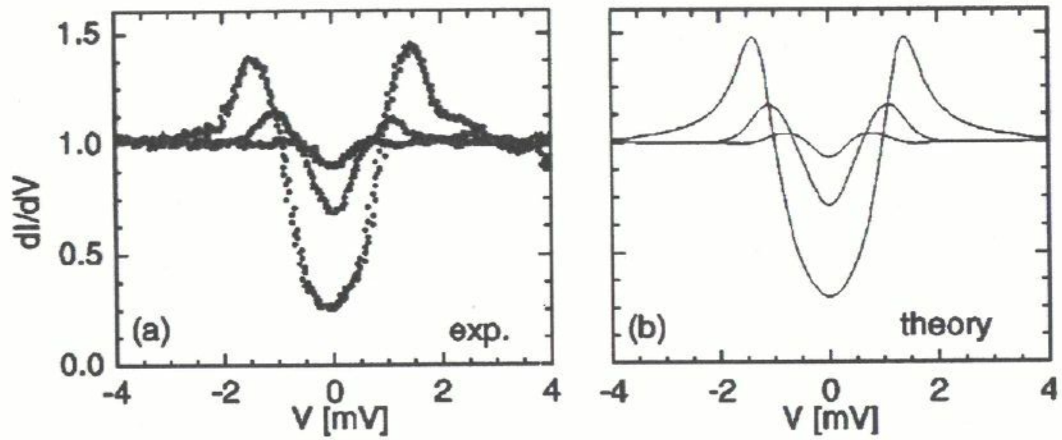
Quasiclassical Theory

of inhomogeneous superconductivity (Eilenberger and Usadel equations)

The Theory requires Ballistic (small samples) or Diffusive (disorder) motion in N

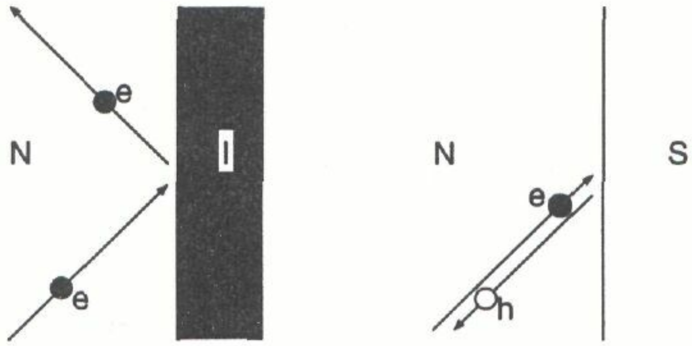


S. Guéron *et. al.*,
Phys. Rev. Lett. **77**, 3025 (1996).

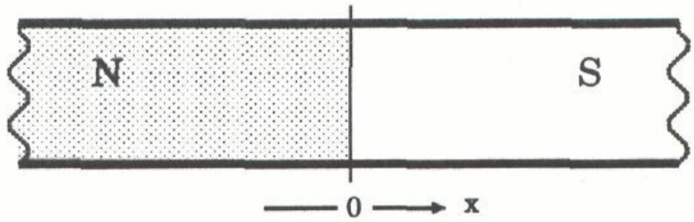


M. Vinet *et. al.*, Phys. Rev. B **63**, 165420 (2001)

Superconducting Proximity Effect



(after C.W.J. Beenakker)
Normal reflection by an insulator (I) versus Andreev reflection by a superconductor (S).



BdG Hamiltonian

$$\begin{pmatrix} \mathcal{H}_0 & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -\mathcal{H}_0^* \end{pmatrix} \begin{pmatrix} \psi_e(\mathbf{r}) \\ \psi_h(\mathbf{r}) \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_e(\mathbf{r}) \\ \psi_h(\mathbf{r}) \end{pmatrix}$$

at $T = 0$:

$$\Delta(\mathbf{r}) = \Delta_0 e^{i\phi} \theta(x)$$

Andreev reflection acts as a boundary condition for N

Scattering approach

Mesoscopic conductance: Kubo versus Landauer

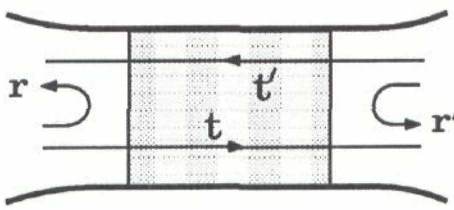
$I = \mathbf{G}_N V$, current = conductance × voltage

Kubo: $\mathbf{G}_N$ (linear response theory)
Landauer: $\mathbf{G}_N = \frac{2e^2}{h} \sum_{n=1}^{N} T_n$
} Fisher & Lee PRB, **23**, 6851 (1981)

zero temperature
zero voltage

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

$$S^\dagger S = 1$$



T_n are the eigenvalues of $t^\dagger t = t'^\dagger t'$

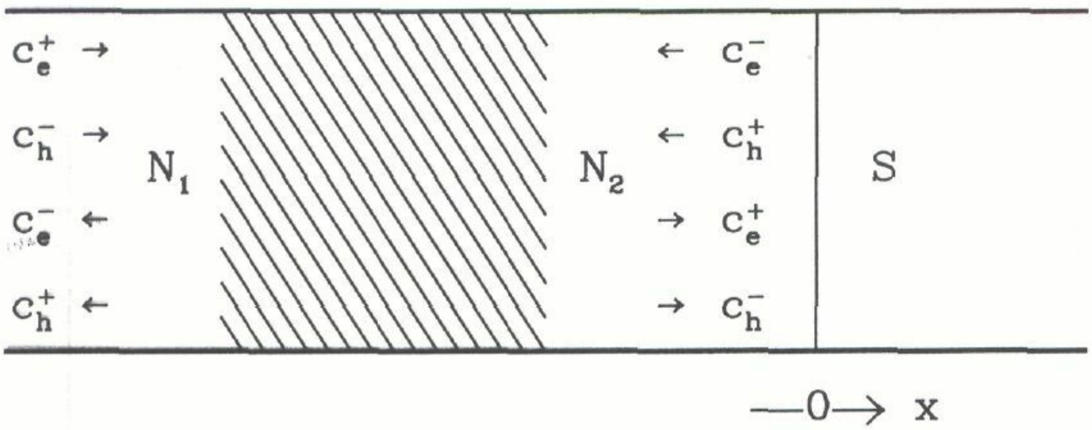
Statistics of $S \mapsto$ Fluctuations of $\mathbf{G}_N$

Scattering approach

Mesoscopic superconductivity:

zero magnetic field
zero temperature
zero voltage

C. W. J. Beenakker: $$\mathrm{G}_{NS} = \frac{4e^2}{h}\sum_{n=1}^{N}\frac{T_n^2}{(2-T_n)^2}$$



$$S_N(\varepsilon) = \begin{pmatrix} S_0(\varepsilon) & 0 \\ 0 & S_0(-\varepsilon)^* \end{pmatrix}$$

$$\begin{pmatrix} c_e^- \\ c_h^+ \end{pmatrix}_{N_2} = \begin{pmatrix} 0 & e^{i\phi - i\phi_A(\varepsilon)} \\ e^{-i\phi - i\phi_A(\varepsilon)} & 0 \end{pmatrix} \begin{pmatrix} c_e^+ \\ c_h^- \end{pmatrix}_{N_2}$$

$\Delta(\mathbf{r}) = \Delta_0 e^{i\phi}\theta(\mathbf{x}),\qquad \phi_{\mathbf{A}}(\varepsilon) = \arccos(\varepsilon/\Delta_0)$

Joint probability density of $S_0(\varepsilon), S_0(-\varepsilon)^*$ $\quad\longmapsto\quad$ Fluctuations of G_{NS}

Scattering approach

to the local density of states

1D: $$H = -\frac{\hbar^2\partial^2}{\partial x^2} + V(x)$$

$$(E + i\eta - H)G(x) = \delta(x) \qquad (*)$$

Expand around $x = 0$:

$$G(x) = c_L(e^{-ikx} + r_L e^{ikx})\theta(-x) + c_R(e^{ikx} + r_R e^{-ikx})\theta(x)$$

Continuity at $x = 0$:

$$c_L(1 + r_L) = c_R(1 + r_R).$$

Eq. $(*)$ provides another relation between c_L and c_R

hence the Green's function:

$$G(0) = \frac{1}{i\hbar v}(1 + r_L)\frac{1}{1 - r_R r_L}(1 + r_R)$$

and the LDOS:

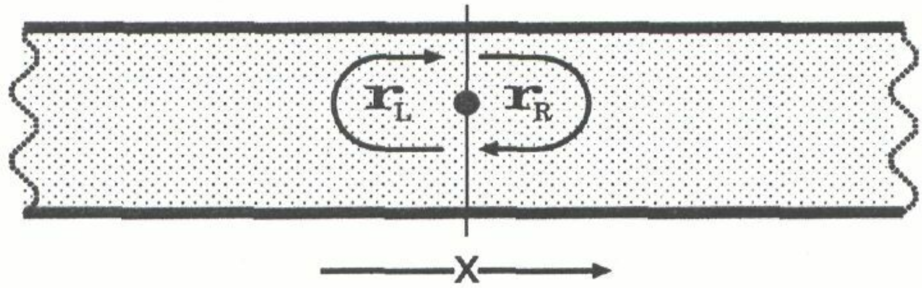
$$n = n_0 \,\mathrm{Re}\,(1 + r_L)\frac{1}{1 - r_R r_L}(1 + r_R)$$

Scattering approach

to LDOS (quasi-1D geometry)

$$\hat{G}(x) = \frac{1}{\sqrt{i\hbar \mathsf{v}}}(1+r_R)\frac{1}{1-r_L r_R}(1+r_L)\frac{1}{\sqrt{i\hbar \mathsf{v}}}$$

$$G_{nm}(x) = \iint_A d\vec{\rho}\, d\vec{\rho}\,' \langle n|\vec{\rho}\rangle\langle\vec{\rho}\,'|m\rangle\langle \mathbf{r}|G^R|\mathbf{r}\rangle$$



$$n(r) = \frac{1}{\pi\hbar}\,\mathrm{Re}\,\mathrm{Tr}\,\hat{P}_\rho \frac{1}{\sqrt{\mathsf{v}}}(1+r_R)\frac{1}{1-r_L r_R}(1+r_L)\frac{1}{\sqrt{\mathsf{v}}}$$

$$\hat{P}_{\rho,nm} = \langle n|\rho\rangle\langle\rho|m\rangle$$

Compare to semiclassical formula   Gutzwiller

$$n(\varepsilon,\mathbf{r}) = n_0 + \mathrm{Im}\sum_\alpha A_\alpha(\varepsilon,\mathbf{r})$$

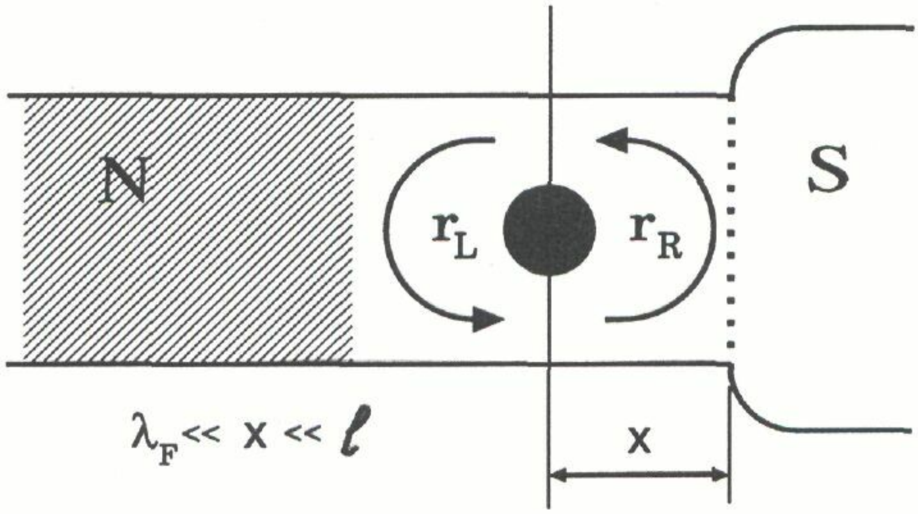
α is a path starting at and returning to $\mathbf{r}$

A_α is the corresponding quantum mechanical amplitude

$$(1+r_R)\frac{1}{1-r_L r_R}(1+r_L)$$
$$= 1 + \sum_{n=1}^{\infty}(r_L r_R)^n + \sum_{n=1}^{\infty}(r_R r_L)^n + \underbrace{\sum_{n=0}^{\infty}(r_L r_R)^n r_L + \sum_{n=0}^{\infty}(r_R r_L)^n r_R}_{\text{oscillate on a scale } \lambda_F}$$

LDOS averaged over a small volume (relevant for a STM measurement) $n(x) = n_0\,\mathrm{Re}\,\frac{1}{N}\mathrm{Tr}\,\frac{1+r_L r_R}{1-r_L r_R}$

NS junction



$$n(x,\varepsilon) = n_0 \,\mathrm{Re}\, \frac{1}{N}\, \mathrm{Tr}\, \frac{1+r_L r_R}{1-r_L r_R}$$

$$r_L = \begin{pmatrix} r_0(\varepsilon) & 0 \\ 0 & r_0(-\varepsilon)^* \end{pmatrix} \qquad r_R = e^{-i\phi_A(\varepsilon)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\phi_A(\varepsilon) = \arccos(\varepsilon/\Delta)$ Andreev phase

$$n(x,\varepsilon) = 1 + \frac{2}{N}\,\mathrm{Re} \sum_{n=1}^{\infty} \mathrm{Tr}\, [r_0(\varepsilon) r_0(-\varepsilon)^*]^n e^{-2in\phi_A}$$

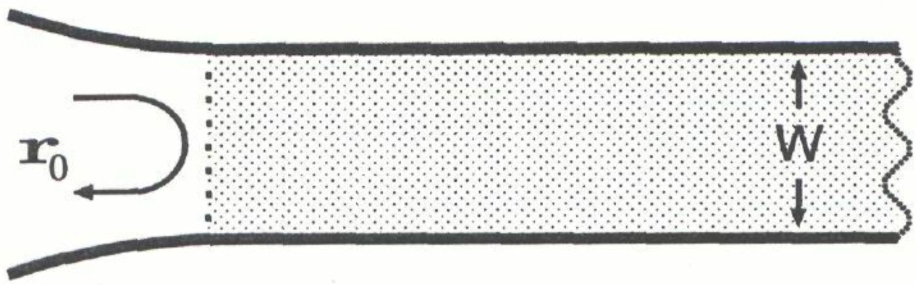
$r_0(\varepsilon) r_0(-\varepsilon)^*$ is a unitary matrix, its eigenvalues are $e^{2i\phi_j}$

Joint probability density $P_\varepsilon(\{\phi_j\})$ comes from DMPK equation

Only one point function $\rho_\varepsilon(\phi)$ is needed to find the mean LDOS

$$\frac{\langle n(x,\varepsilon)\rangle}{n_0} = \pi \rho_\varepsilon(\phi_A) \qquad \varepsilon < \Delta$$

Random Matrix Theory



Consider matrix correlator $r_0(\varepsilon)r_0(-\varepsilon)^*$ with eigenvalues $e^{2i\phi_j}$

$$r_0(\varepsilon)r_0(-\varepsilon)^* \underset{\text{TRS}}{=} r_0(\varepsilon)r_0(-\varepsilon)^\dagger \underset{\varepsilon=i\omega}{\rightarrow} r_0(i\omega)r_0(i\omega)^\dagger$$

Eigenvalues of $r_0(i\omega)r_0(i\omega)^\dagger$ are reflection probabilities $0 < R_j < 1$

The convenient parametrization is

$$R_j = \frac{\sigma_j}{\sigma_j + 2(N+1)\omega\tau_s}, \qquad \sigma_j \in (0, \infty), \quad \tau_s = \ell/v_F.$$

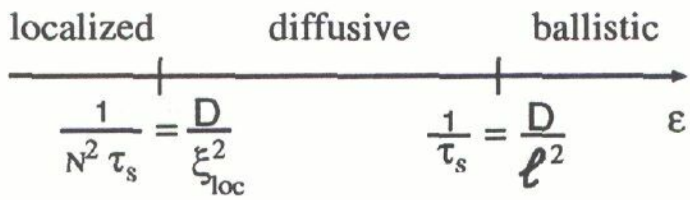
The joint probability density follows from DMPK equation.
For infinite wire it is given by the Laguerre ensemble (LOE)

$$P_{i\omega}(\{\sigma_j\}) = c_N \prod_{j=1}^{N} e^{-\sigma_j/4} \prod_{j<k} |\sigma_k - \sigma_j|$$

Scales

How $\rho_\varepsilon(\phi)$ looks like

$$\langle n\rangle = \pi\rho_\varepsilon(\phi_A), \quad \varepsilon < \Delta$$



$$\rho_\varepsilon(\phi)|_{N=\infty} = \frac{1}{\pi\sin^2\phi}\ \mathrm{Im}\sqrt{(\varepsilon\tau_s)^2 + i\varepsilon\tau_s(1-e^{-2i\phi})}$$

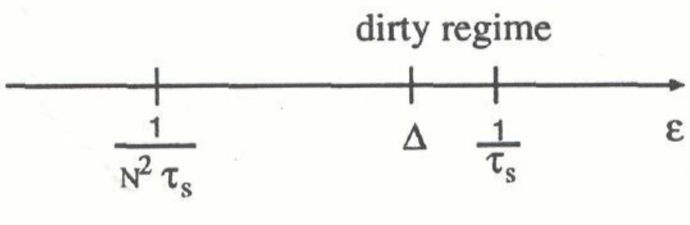
Titov, Beenakker
PRL 85, 3388 (2000)

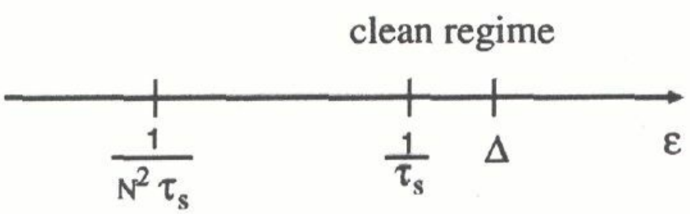
$$\rho_\varepsilon(\phi)|_{N=1} = \frac{\varepsilon\tau_s}{\pi}\int_0^\infty \frac{\exp(-\varepsilon\tau_s t)}{t^2\sin^2\phi - t\sin 2\phi + 1}dt$$

Berezinskii, Gor'kov
JETP 50, 1209 (1979)

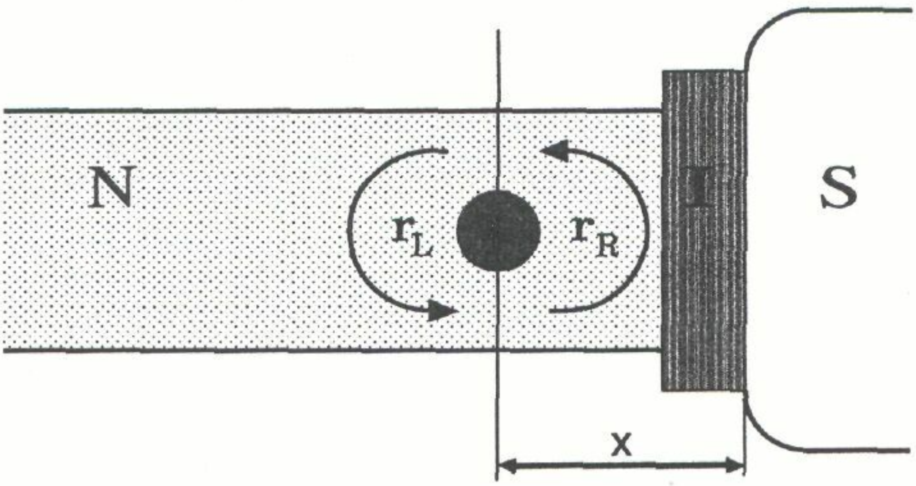
The dependence on Δ appears only due to Andreev phase

Two regimes are distinguished





NIS junction



$$n(x,\varepsilon) = n_0 \,\mathrm{Re}\, \frac{1}{N}\, \mathrm{Tr}\, \frac{1+r_L r_R}{1-r_L r_R}$$

$$r_L = \begin{pmatrix} r_0(\varepsilon) & 0 \\ 0 & r_0(-\varepsilon)^* \end{pmatrix} = \begin{pmatrix} u_0 & 0 \\ 0 & u_0^* \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} u_0^T & 0 \\ 0 & u_0^\dagger \end{pmatrix}$$

$$r_R = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} e^{i\chi}\cos\theta & -ie^{i\chi}\sin\theta \\ -ie^{i\chi}\sin\theta & e^{i\chi}\cos\theta \end{pmatrix} \begin{pmatrix} u^T & 0 \\ 0 & u^\dagger \end{pmatrix}$$

$$\sin\theta = \Gamma\left[(1-e^{2i\phi_A}(1-\Gamma))(1-e^{-2i\phi_A}(1-\Gamma))\right]^{-1/2}$$

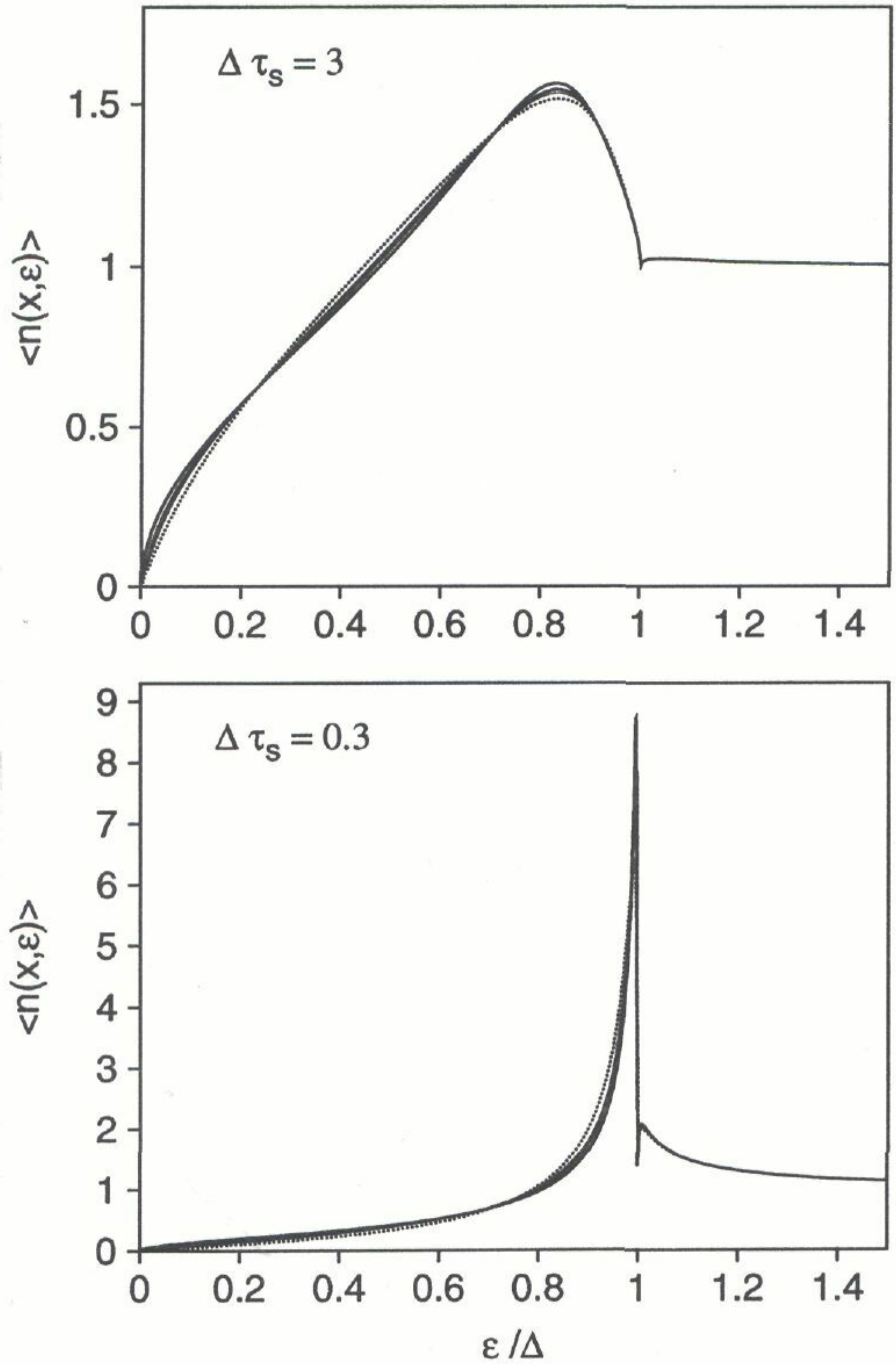
$$e^{2i\chi} = (1-\Gamma-e^{2i\phi_A})\left[1-e^{2i\phi_a}(1-\Gamma)\right]^{-1}$$

u_0 is a random unitary matrix

ϕ is a diagonal matrix of eigenphases distributed with the density $\rho_\varepsilon(\phi)$

Γ is a diagonal matrix of the transmission probabilities $0 < \Gamma_j < 1$.

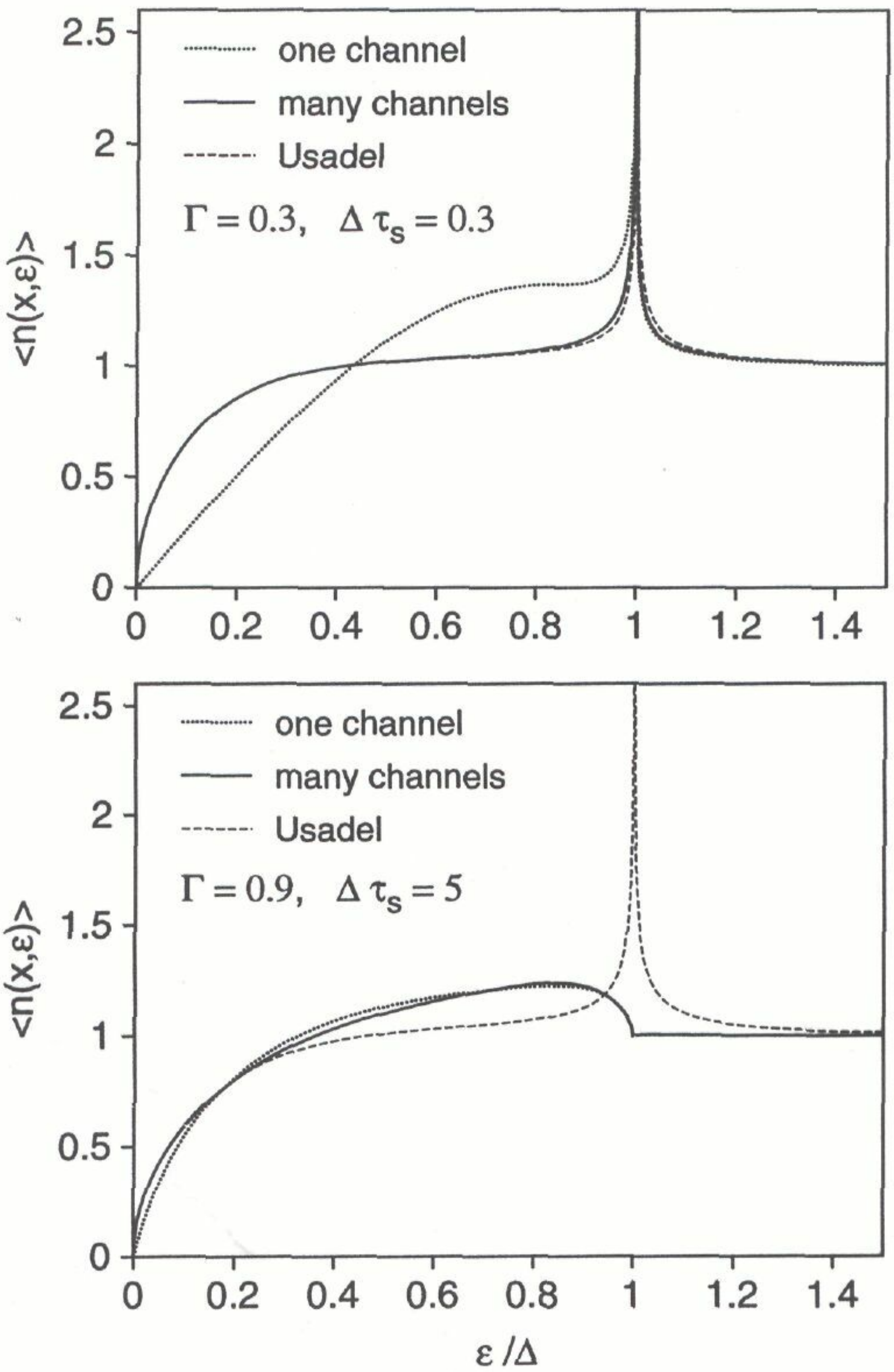
Plots



$$\langle n(x,\varepsilon)\rangle = (\pi/4)(N+1)\varepsilon\tau_s, \quad \varepsilon \ll (N^2\tau_s)^{-1}$$

$$\langle n(x,\varepsilon)\rangle|_{N=\infty} = \mathrm{Re}\sqrt{-i\varepsilon\tau_s}, \qquad \varepsilon \ll 1/\tau_s$$

with a barrier



$$\langle n(x,\varepsilon)\rangle = (\pi/4)(N+1)\varepsilon\tau_s(2-\Gamma)/\Gamma, \quad \varepsilon \ll (N^2\tau_s)^{-1}$$

$$\langle n(x,\varepsilon)\rangle|_{N=\infty} = \mathrm{Re}\sqrt{-i\varepsilon\tau_s}(2-\Gamma)/\Gamma, \quad \varepsilon \ll 1/\tau_s$$

Outcome

- We developed a scattering approach to the LDOS

- We studied in some detail the effects of localization on the shape of the LDOS in NS and NIS mesoscopic junctions