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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 3

Algebra of conformal Killing vector fields

Definition 1. *Let $(M^n, g_{ij}(\vec{x})$ be a Riemannian manifold. Vector field $\vec{v}(\vec{x})$ is called a Killing vector field, if $(L_{\vec{v}}g)_{ij} = 0$.*

Definition 2. *Let M^n be a manifold equipped with a conformal structure, $g_{ij}(\vec{x})$ be a Riemannin metric representing the conformal class of M^n. Vector field $\vec{v}(\vec{x})$ is called a conformal Killing vector field, if $(L_{\vec{v}}g)_{ij} = \lambda(\vec{x})g_{ij}(\vec{x})$ for some scalar function $\lambda(\vec{x})$.*

It is easy to check that Definition 2 is correct. Assume that we have two different metrics $g_{ij}(\vec{x})$, $\widetilde{g}_{ij}(\vec{x})$ representing the same conformal class:

$$\tilde{g}_{ij}(\vec{x}) = \alpha(\vec{x})g_{ij}(\vec{x}), \quad \alpha(\vec{x}) \in \mathbb{R}, \quad \alpha(\vec{x}) > 0.$$

Then

$$(L_{\vec{v}}\widetilde{g})_{ij}(\vec{x}) = L_{\vec{v}}(\alpha(\vec{x})g_{ij}(\vec{x})) = L_{\vec{v}}\alpha(\vec{x})\ g_{ij}(\vec{x}) + \alpha(\vec{x})L_{\vec{v}}g_{ij}(\vec{x}) =$$

$$= L_{\vec{v}}\alpha(\vec{x})\ g_{ij}(\vec{x}) + \alpha(\vec{x})\lambda(\vec{x})g_{ij}(\vec{x}) = \frac{L_{\vec{v}}\alpha(\vec{x})}{\alpha(\vec{x})}\widetilde{g}_{ij}(\vec{x}) + \alpha(\vec{x})\lambda(\vec{x})\alpha^{-1}(\vec{x})\widetilde{g}_{ij}(\vec{x}) =$$

$$= \widetilde{\lambda}(\vec{x})\widetilde{g}_{ij}(\vec{x}),$$

where

$$\widetilde{\lambda}(\vec{x}) = \frac{L_{\vec{v}}\alpha(\vec{x})}{\alpha(\vec{x})} + \lambda(\vec{x}).$$

From the formula for the Lie derivative it follows immediately that:

Theorem 1. *$V(\vec{x})$ is a generator of an infinitesimal conformal transformation, if*

$$v^\alpha \frac{\partial g_{ij}}{\partial x^\alpha} + \frac{\partial v^\alpha}{\partial x^i} g_{\alpha j} + \frac{\partial v^\alpha}{\partial x^j} g_{i\alpha} = \lambda(\vec{x})g_{ij}.$$

Let me recall, that the situation with conformal maps is completely different for $n = 2$ and $n \geq 3$.

1) Let $n = 2$. Then we can treat $\mathbb{R}^2$ as $\mathbb{C}^1$. Let U be an open subset $U \subset \mathbb{C}^1$, $f(z)$ be a locally invertible map $U \to \mathbb{C}^1$. Then this map is conformal iff $f(z)$ is either **holomorphic** or **antiholomorphic**. Holomorphic maps preserve the orientation, antiholomorphic maps reverse the orientation. The space of such maps is **infinite-dimensional**.

   The situation changes if we consider **global** conformal transformations.

   Theorem 2. *Let $S^2 = \mathbb{C}P^1$ be the Riemann sphere. Then the group of globally defined invertible conformal maps coincides with the 2-dimensional Möbius group.*

2) For $n \geq 3$ the following statement takes place:

Theorem 3. *Let $M^n, n \geq 3$ be a manifold equipped with a conformal structure, $U \subset M^n$. Then the space of conformal maps $U \to M^n$ is* ***finite dimensional*** *and its dimension is not higher than the dimension of the Mobius group.*

A special case of this theorem is known as **Liouville theorem.**

Theorem 4. *Let $M^n = \mathbb{R}^n$, and conformal structure is generated by the standard Euclidean metric $g_{ij}(\vec{x}) = \delta_{ij}$. Then any conformal map $U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ coincides with the restriction of some Möbius map $\mathbb{R}^n \to \mathbb{R}^n$ to U.*

Proof of Liouville theorem can be found in the book "Modern geometry" by Dubrovin, Novikov, Fomenko.

We do not provide a proof of these Theorems in our course. But to explain it, we proof an infinitesimal version of the Liouville theorem for $\mathbb{R}^3$. As we explained above, a vector field $\vec{v}(\vec{x})$ generates a one-parametric family of conformal maps iff $\vec{v}(\vec{x})$ is conformal Killing, i.e.

$$L_{\vec{v}} g(\vec{x})_{ij} = \lambda(\vec{x}) g(\vec{x})_{ij},$$

for some real-valued function $\lambda(\vec{x})$.

Theorem 5. *Let $U \subset \mathbb{R}^3$ be an open subset of $\mathbb{R}^3$ equipped with conformal structure generated by the standard Euclidean metric $g(\vec{x})_{ij} = \delta_{ij}$. Then the algebra of conformal Killing vector fields coincides with the Lie algebra of the Möbius group.*

Remark 1. *In fact, this Theorem is true for any $n \geq 3$, and our proof works for any $n \geq 3$ after minor modifications. But to avoid unnecessary technical complications we consider in our lectures only the case $n = 3$.*

Proof. Calculating the Lie derivative for $g(\vec{x})_{ij} = \delta_{ij}$ we immediately obtain

$$L_{\vec{v}} \delta_{ij} = \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i},$$

therefore

$$2\frac{\partial v^i}{\partial x^i} = \lambda(\vec{x}),$$

$$\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} = 0, \quad \text{for all} \quad i \neq j.$$

$$\frac{\partial v^i}{\partial x^i} = \frac{\partial v^j}{\partial x^j}, \qquad \text{for all} \quad i, j, \tag{1}$$

$$\frac{\partial v^i}{\partial x^j} = -\frac{\partial v^j}{\partial x^i}, \qquad \text{for all} \quad i \neq j. \tag{2}$$

Let us remark that for $n = 2$ we get

$$\begin{cases} \frac{\partial v^1}{\partial x^1} = \frac{\partial v^2}{\partial x^2}, \\ \frac{\partial v^2}{\partial x^1} = -\frac{\partial v^1}{\partial x^2}. \end{cases} \tag{3}$$

System (3) coincides with the Cauchy-Riemann equations. Therefore conformal Killing vector fields in $\mathbb{R}^2 = \mathbb{C}^1$ are exactly holomorphic vector fields.

Let us return to the $n = 3$ case. Denote the coordinates in $\mathbb{R}^3$ by (x, y, z). Let us calculate the conformal Killing vector fields algebra step by step.

Step 1. From (1), (2) it follows that vector field $\vec{v}(x)$ generates an infinitesimal conformal transformation iff:

$$\frac{\partial v^1}{\partial x} = \frac{\partial v^2}{\partial y} = \frac{\partial v^3}{\partial z}, \tag{4}$$

$$\frac{\partial v^1}{\partial y} = -\frac{\partial v^2}{\partial x}, \tag{5}$$

$$\frac{\partial v^1}{\partial z} = -\frac{\partial v^3}{\partial x}, \tag{6}$$

$$\frac{\partial v^2}{\partial z} = -\frac{\partial v^3}{\partial y}, \tag{7}$$

Step 2. It is easy to check that the following vector fields satisfy equations (4)-(7):

1) generators of translations

$$P_1 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad P_3 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{8}$$

2) generators of rotations

$$M_{12} : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}, \quad M_{23} : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}, \quad M_{31} : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}, \tag{9}$$

3) generator of dilations

$$D : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \tag{10}$$

4) conformal generators

$$K_1 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} x^2 - y^2 - z^2 \\ 2xy \\ 2xz \end{bmatrix}, K_2 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 2xy \\ y^2 - x^2 - z^2 \\ 2yz \end{bmatrix}, K_3 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 2xz \\ 2yz \\ z^2 - x^2 - y^2 \end{bmatrix}. \tag{11}$$

Step 3. Using (5), (7), we obtain

$$\frac{\partial^2 v^1}{\partial y \partial z} = -\frac{\partial^2 v^2}{\partial x \partial z} = \frac{\partial^2 v^3}{\partial x \partial y}$$

Using (6), we obtain

$$\frac{\partial^2 v^1}{\partial y \partial z} = -\frac{\partial^2 v^3}{\partial x \partial y},$$

therefore

$$\frac{\partial^2 v^1}{\partial y \partial z} = 0, \quad \text{and} \quad v^1 = F_1(x,y) + F_2(x,z). \tag{12}$$

We have

$$\frac{\partial v^1}{\partial x} = \frac{\partial v^2}{\partial y},$$

therefore

$$\frac{\partial^2 v^1}{\partial x^2} = \frac{\partial v^2}{\partial x \partial y} = -\frac{\partial^2 v^1}{\partial y^2},$$

and

$$\frac{\partial^2 v^1}{\partial x^2} = -\frac{\partial^2 v^1}{\partial z^2}.$$

We see that

$$\frac{\partial^2 v^1}{\partial y^2} - \frac{\partial^2 v^1}{\partial z^2} = 0, \tag{13}$$

By comparing (12), (13) and using analogous arguments for v^2, v^3 we obtain:

$$v^1 = f_{00}(x) + f_{10}(x)y + f_{01}(x)z + f_{11}(x)[y^2 + z^2], \tag{14}$$
$$v^2 = g_{00}(y) + g_{10}(y)x + g_{01}(y)z + g_{11}(y)[x^2 + z^2], \tag{15}$$
$$v^3 = h_{00}(z) + h_{10}(z)x + h_{01}(z)y + h_{11}(z)[x^2 + y^2]. \tag{16}$$

By substituting (14), (15) into (5), we obtain:

$$f_{10}(x) + 2f_{11}(x)y = -g_{10}(y) - 2g_{11}(y)x, \tag{17}$$

Therefore the functions $f_{10}(x)$, $f_{11}(x)$ are linear in x, $g_{10}(y)$, $g_{11}(y)$ are linear in y. Using equations (6), (7), we obtain:

$$v^1 = f_{00}(x) + (f_{10}^0 + f_{10}^1 x)y + (f_{01}^0 + f_{01}^1 x)z + (f_{11}^0 + f_{11}^1 x)[y^2 + z^2], \tag{18}$$
$$v^2 = g_{00}(y) + (g_{10}^0 + g_{10}^1 y)x + (g_{01}^0 + g_{01}^1 y)z + (g_{11}^0 + g_{11}^1 y)[x^2 + z^2], \tag{19}$$
$$v^3 = h_{00}(z) + (h_{10}^0 + h_{1o}^1 z)x + (h_{01}^0 + h_{01}^1 z)y + (h_{11}^0 + h_{11}^1 z)[x^2 + y^2]. \tag{20}$$

From (17) and analogous equations for (6), (7) if follows that

$$f_{11}^1 = -g_{11}^1, \quad f_{11}^1 = -h_{11}^1, \quad g_{11}^1 = -h_{11}^1,$$

$$f_{10}^1 = -2g_{11}^0, \quad f_{01}^1 = -2h_{11}^0, \quad g_{10}^1 = -2f_{11}^0, \quad g_{01}^1 = -2h_{11}^0, \quad h_{10}^1 = -2f_{11}^0, \quad h_{01}^1 = -2g_{11}^0,$$

therefore

$$f_{11}^1 = g_{11}^1 = h_{11}^1 = 0,$$

and

$$v^1 = f_{00}(x) + (f^0_{10} - 2g^0_{11}x)y + (f^0_{01} - 2h^0_{11}x)z + f^0_{11}[y^2 + z^2], \tag{21}$$
$$v^2 = g_{00}(y) + (g^0_{10} - 2f^0_{11}y)x + (g^0_{01} - 2h^0_{11}y)z + g^0_{11}[x^2 + z^2], \tag{22}$$
$$v^3 = h_{00}(z) + (h^0_{10} - 2f^0_{11}z)x + (h^0_{01} - 2g^0_{11}z)y + h^0_{11}[x^2 + y^2]. \tag{23}$$

By adding generators (8) - (10) we can cancel constant and linear terms at the point $x = y = z = 0$, and without loss of generality we obtain:

$$v^1 = f_{00}(x) - 2g^0_{11}xy - 2h^0_{11}xz + f^0_{11}[y^2 + z^2], \tag{24}$$
$$v^2 = g_{00}(y) - 2f^0_{11}yx - 2h^0_{11}yz + g^0_{11}[x^2 + z^2], \tag{25}$$
$$v^3 = h_{00}(z) - 2f^0_{11}zx - 2g^0_{11}zy + h^0_{11}[x^2 + y^2], \tag{26}$$
$$f_{00}(0) = g_{00}(0) = h_{00}(0) = 0. \tag{27}$$

From (4) it follows that

$$f_{00}(x) = -f^0_{11}x^2, \quad g_{00}(y) = -g^0_{11}y^2, \quad h_{00}(z) = -h^0_{11}z^2.$$

We proved the following Theorem:

Theorem 6. *The Lie algebra of vector fields generating local conformal transformations of $\mathbb{R}^3$ has the following 10-dimensional basis $P_1, P_2, P_3, M_{12}, M_{23}, M_{31}, D, K_1, K_2, K_3$, where:*

$$P_1 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad P_3 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$M_{12} : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}, \quad M_{23} : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}, \quad M_{31} : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix},$$

$$D : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad K_1 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} x^2 - y^2 - z^2 \\ 2xy \\ 2xz \end{bmatrix},$$

$$K_2 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 2xy \\ y^2 - x^2 - z^2 \\ 2yz \end{bmatrix}, \quad K_3 : \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 2xz \\ 2yz \\ z^2 - x^2 - y^2 \end{bmatrix}.$$

Therefore, we proved that these 10 vector fields form a full basis of solutions and we also discovered that it is finite-dimensional.