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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 6

Weil and Cotton tensors

Definition 1. *The Riemannian manifold* $(M^n, g_{ij}(\vec{x})$ *is called flat, if near each point* x_0 *there exists a coordinate system* $\tilde{x}^1, \ldots, \tilde{x}^n$ *such that in the new coordinates*

$$\tilde{g}_{ij}(x) = \delta_{ij}. \tag{1}$$

is flat.

Definition 2. *The conformal manifold* $(M^n, \{g_{ij}(\vec{x})\}$ *is called conformally flat, if near each point* x_0 *there exists a coordinate system* $\tilde{x}^1, \ldots, \tilde{x}^n$ *such that in the new coordinates the Riemannian metric* $g_{ij}(\vec{x})$ *representing the conformal structure has the form*

$$\tilde{g}_{ij}(x) = \lambda(x)\delta_{ij}. \tag{2}$$

is flat.

It is easy to check that the definition of conformal flatness does not depend on the choice of the Riemannian metric representing the conformal class.

Example 1. *In the coordinates of the stereographic projection the natural metric on* S^n *has the form*

$$ds^2 = 4\,\frac{(dx^1)^2 + \ldots + (dx^n)^2}{1 + (x^1)^2 + \ldots + (x^n)^2},$$

therefore S^n *equipped with the standard conformal structure is conformally flat. Analogously, Lobachevsky spaces* L^n *are conformally flat for all* n*:*

$$ds^2 = 4\,\frac{(dx^1)^2 + \ldots + (dx^n)^2}{1 - (x^1)^2 + \ldots - (x^n)^2},$$

Of course, neither S^n *nor* L^n *are flat.*

Let us recall some basic definitions from the Riemannian geometry.

1. Christoffel symbols:
$$\Gamma^k_{ij} = \frac{1}{2} g^{k\alpha}\left[\partial_i g_{\alpha j} + \partial_j g_{i\alpha} - \partial_\alpha g_{ij}\right]$$

2. Riemann curvature tensor is defined by
$$(\nabla_k \nabla_l - \nabla_l \nabla_k) T^i = R^i{}_{jkl} T^j$$
Here T^i is a vector field.

   Corollary:
$$(\nabla_k \nabla_l - \nabla_l \nabla_k)\alpha_q = -R^p{}_{qkl}\alpha_p,$$
where α_q is a one-form. Explicit expression:
$$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{k\alpha}\Gamma^\alpha_{lj} - \Gamma^i_{l\alpha}\Gamma^\alpha_{kj}$$

By lowering the first index we obtain:

$$R_{ijkl} = g_{i\alpha} R^{\alpha}{}_{jkl}$$

It has the following symmetries:

a) $R_{ijlk} = -R_{ijkl}$,

b) $R_{jikl} = -R_{ijkl}$,

c) $R_{klij} = R_{ijkl}$,

3. Biancki algebraic identities:

$$R_{sijk} + R_{sjki} + R_{skij} = 0.$$

Let us remark, that for $n = 2, 3$ these identities are trivial, and for $n \geq 4$ they are non-trivial.

4. Ricci tensor:

$$R_{kl} = R^{\alpha}{}_{k\alpha l} = g^{\alpha\beta} R_{\alpha k\beta l}$$

Ricci tensor is symmetric:

$$R_{kl} = R_{lk}.$$

5. Scalar curvature tensor:

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

Let us recall that

Theorem 1. *A Riemannian manifold M^n with the metric g_{ij} admits flat coordinates such that $g_{ij} = \delta_{ij}$ iff the curvature tensor is identically zero:*

$$R^{i}{}_{jkl} \equiv 0, \quad \textit{for all} \;\; i, j, k, l.$$

Definition 3. *A Riemannian manifold M^n with the metric g_{ij} is callled* ***coformally flat*** *if there exists a scalar real function ω such that the Riemann metric $\tilde{g}_{ij} = e^{2\omega} g_{ij}$ is flat.*

How to check if a Riemannian manifold is conformally flat. The answer is provided by

Theorem 2. *1. If $n = 2$ a metric g_{ij} is alway conformally flat,*

2. if $n = 3$ a metric is conformally flat iff its Cotton tensor is identically equal to zero: $C_{ikj} \equiv 0$;

3. if $n \geq 4$ a metric is conformally flat iff its Weyl tensor is identically equal to zero: $W_{ikjl} \equiv 0$.

I dot plan to provide a complete proof. The first part will be proved later in the 2-d part of the course. At the next lecture I plan to define these tensors and prove that for $n \geq 4$ the Weil tensor in conformal invariant, and the variation of the Cotton tensor with respect to conformal changes of metric is proportional to the Weyl tensor. If $n = 3$, the Weil tensor vanished identically, therefore the Cotton tensor is conformally invariant.

Let us define some basic objects

1. Schouten tensor:
$$P_{ij} = \frac{1}{n-2}\left[R_{ij} - \frac{1}{2(n-1)} R\, g_{ij}\right] \tag{3}$$

2. Weyl tensor:
$$W_{ijkl} = R_{ijkl} - \left[g_{ik}P_{jl} + g_{jl}P_{ik} - g_{il}P_{jk} - g_{jk}P_{il}\right] \tag{4}$$
$$W^i_{\ jkl} = R^i_{\ jkl} - \left[\delta^i_k P_{jl} + g_{jl}g^{i\alpha}P_{\alpha k} - \delta^i_l P_{jk} - g_{jk}g^{i\alpha}P_{\alpha l}\right] \tag{5}$$

3. Cotton tensor:
$$C_{jkl} = (n-2)\left[\nabla_l P_{jk} - \nabla_k P_{jl}\right]. \tag{6}$$

Let us also recall the standard definition:
$$\partial^k = g^{k\alpha}\partial_\alpha, \quad \nabla^k = g^{k\alpha}\nabla_\alpha.$$

We shall also use:
$$\nabla_k g_{ij} = 0.$$

Consider the following change of metric:
$$\tilde{g}_{ij} = e^{2\omega(x)} g_{ij} \tag{7}$$

Let us calculate step by step how this change of metric within the same conformal class affects the basic differential-geometric objects:

Theorem 3. *Let M^n, $g_{ij}(x)$ be a Riemannian manifold. If we introduce a new metric $\tilde{g}_{ij}(x)$ using formula (7), then we have the following transformation rules:*

1.
$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij},$$
where
$$S^k_{ij} = \delta^k_i\, \partial_j\omega + \delta^k_j\, \partial_i\omega - g_{ij}\, \partial^k\omega$$

2.
$$\widetilde{R}^i_{\ jkl} = R^i_{\ jkl} + \nabla_k S^i_{lj} - \nabla_l S^i_{kj} + S^i_{k\alpha}S^\alpha_{lj} - S^i_{l\alpha}S^\alpha_{kj} =$$
$$= R^i_{\ jkl} + \delta^i_l\, \nabla_k\partial_j\omega - \delta^i_k\, \nabla_l\partial_j\omega + g_{jk}\, \nabla_l\partial^i\omega - g_{jl}\, \nabla_k\partial^i\omega +$$
$$+ \delta^i_k\, \partial_j\omega\partial_l\omega - \delta^i_l\, \partial_j\omega\partial_k\omega + g_{jl}\, \partial^i\omega\partial_k\omega - g_{jk}\, \partial^i\omega\partial_l\omega + \left[\delta^i_l g_{jk} - \delta^i_k g_{jl}\right]\left[\partial^\alpha\omega\partial_\alpha\omega\right]$$

3.

$$\widetilde{R}_{jl} = R_{jl} - g_{jl}\nabla^\alpha\partial_\alpha\omega - (n-2)\nabla_l\partial_j\omega + (n-2)\partial_j\omega\partial_l\omega - (n-2)g_{jl}\partial^\alpha\omega\partial_\alpha\omega.$$

4.

$$\widetilde{R} = e^{-2\omega}\big[R - 2(n-1)\nabla^\alpha\partial_\alpha\omega - (n-1)(n-2)\partial^\alpha\omega\partial_\alpha\omega\big]$$

Proof. 1. Let us now consider the change of metric in Christoffel symbols. Since

$$\partial_j\tilde{g}_{\alpha k} = 2[\partial_j\omega]e^{2\omega}g_{\alpha k} + e^{2\omega}\partial_j g_{\alpha k} = e^{2\omega}[\partial_j g_{\alpha k} + 2\partial_j\omega\tilde{g}_{\alpha k}]$$

we have

$$\widetilde{\Gamma}^i_{jk} = \frac{1}{2}e^{-2\omega}g^{i\alpha}\left[e^{2\omega}\partial_j g_{\alpha k} + 2\partial_j\omega\tilde{g}_{\alpha k} + e^{2\omega}\partial_k g_{j\alpha} + 2\partial_k\omega\tilde{g}_{j\alpha} - e^{2\omega}\partial_\alpha g_{jk} - 2\partial_\alpha\omega\tilde{g}_{jk}\right]$$

and, similarly, taking into account the definition $\partial^k := g^{k\alpha}\partial_\alpha$

$$S^i_{jk} = g^{i\alpha}\left[\partial_j\omega\tilde{g}_{\alpha k} + \partial_k\omega\tilde{g}_{j\alpha} - \partial_\alpha\omega\tilde{g}_{\alpha k}\right] = \delta^i_k\partial_j\omega + \delta^i_j\partial_k\omega - \partial^i\omega g_{jk}.$$

2. For the Riemann tensor we obtain:

$$\widetilde{R}^i{}_{jkl} = \partial_k\Gamma^i_{lj} + \partial_k S^i_{lj} - \partial_l\Gamma^i_{kj} - \partial_l S^i_{kj} + [\Gamma^i_{k\alpha} + S^i_{k\alpha}][\Gamma^\alpha_{lj} + S^\alpha_{lj}] - [\Gamma^i_{l\alpha} + S^i_{l\alpha}][\Gamma^\alpha_{kj} + S^\alpha_{kj}] =$$

$$= R^i_{jkl} + \partial_k S^i_{lj} - \partial_l S^i_{kj} + \Gamma^i_{k\alpha}S^\alpha_{lj} + \Gamma^\alpha_{lj}S^i_{k\alpha} + S^i_{k\alpha}S^\alpha_{lj} - \Gamma^i_{l\alpha}S^\alpha_{kj} - \Gamma^\alpha_{kj}S^i_{l\alpha} - S^i_{l\alpha}S^\alpha_{kj} -$$
$$-\Gamma^\alpha_{kl}S^i_{\alpha j} + \Gamma^\alpha_{lk}S^i_{lj} = R^i_{jkl} + \nabla_k S^i_{lj} - \nabla_l S^i_{kj} + S^i_{k\alpha}S^\alpha_{lj} - S^i_{l\alpha}S^\alpha_{kj}.$$

3. For the Ricci tensor we obtain:

$$\widetilde{R}_{jl} = \widetilde{R}^i_{jil} = R_{jl} + \nabla_i S^i_{lj} - \nabla_l S^i_{ij} + S^i_{i\alpha}S^\alpha_{lj} - S^i_{l\alpha}S^\alpha_{ij}.$$

In order to compute it, we have to consider

$$\nabla_i S^i_{lj} - \nabla_l S^i_{ij} + S^i_{i\alpha}S^\alpha_{lj} - S^i_{l\alpha}S^\alpha_{ij},$$

where

$$S^i_{i\alpha} = \delta^i_\alpha\partial_i\omega + \delta^i_i\partial_\alpha\omega - g_{\alpha i}\partial^i\omega = \partial_\alpha\omega + n\partial_\alpha\omega - \partial_\alpha\omega = n\partial_\alpha\omega.$$

So, we get

$$\nabla_i S^i_{lj} - \nabla_l S^i_{ij} = \nabla_i[\delta^i_l\partial_j\omega + \delta^i_j\partial_l\omega - g_{lj}\partial^i\omega] - n\nabla_l\partial_j\omega =$$

$$= \nabla_l\partial_j\omega + \nabla_j\partial_l\omega - g_{lj}\nabla_i\partial^i\omega - n\nabla_l\partial_j\omega$$

and since

$$\nabla_l\partial_j\omega = \partial_l\partial_j\omega - \Gamma^\alpha_{lj}\partial_\alpha\omega = \nabla_j\partial_l\omega,$$

we have

$$\nabla_i S^i_{lj} - \nabla_l S^i_{ij} = -g_{lj}\nabla_\alpha\partial^\alpha\omega + (2-n)\nabla_l\nabla_j\omega.$$

Then

$$S^i_{i\alpha}S^\alpha_{lj}-S^i_{l\alpha}S^\alpha_{ij} = n\partial_\alpha\omega[\delta^\alpha_l\partial_j\omega+\delta^\alpha_j\partial_l\omega-g_{lj}\partial^\alpha\omega]-[\delta^i_l\partial_\alpha\omega+\delta^i_\alpha\partial_l\omega-g_{l\alpha}\partial^i\omega][\delta^\alpha_i\partial_j\omega+\delta^\alpha_j\partial_i\omega-g_{ij}\partial^\alpha\omega] =$$

$$= 2n\partial_l\omega\partial_j\omega-ng_{lj}\partial_\alpha\omega\partial^\alpha\omega-\partial_l\omega\partial_j\omega-\partial_l\omega\partial_j\omega+g_{jl}\partial_\alpha\omega\partial^\alpha\omega-n\partial_j\omega\partial_l\omega-\partial_j\omega\partial_l\omega+$$
$$+\partial_l\omega\partial_j\omega+\partial_l\omega\partial_j\omega+g_{lj}\partial^i\omega\partial_i\omega-\partial_j\omega\partial_l\omega = (n-2)(\partial_l\omega\partial_j\omega-g_{jl}\partial_\alpha\omega\partial^\alpha\omega).$$

4. For the scalar curvature we obtain:

$$e^{2\omega}\widetilde{R} = R-g^{jl}g_{jl}\nabla^\alpha\partial_\alpha\omega-(n-2)g^{jl}\nabla_l\partial_j\omega+(n-2)g^{jl}\partial_j\omega\partial_l\omega-(n-2)g^{il}g_{jl}\partial_\alpha\omega\partial^\alpha\omega =$$

$$= R-n\nabla^\alpha\partial_\alpha\omega-(n-2)\partial^\alpha\partial_\alpha\omega+(n-2)\partial^\alpha\omega\partial_\alpha\omega-n(n-2)\partial^\alpha\omega\partial_\alpha\omega =$$
$$= R-2(n-1)\nabla_\alpha\partial_\alpha\omega-(n-1)(n-2)\partial^\alpha\omega\partial_\alpha\omega.$$

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