Grinevich Petr Georgievich

Lecture notes

Conformal geometry and Riemann surfaces

Lecture notes were prepared by the student of the mathematics and mechanics faculty
Raev Aleksey Alekseevich

Lecture 7

Weil tensor is conformally invariant

Let $(M^n, \{g_{ij}(\vec{x})\})$ be a conformal manifold. To define the Weil tensor it is necessary to fix a Riemannian metric representing this conformal structure. Let us show that the Weil tensor is well-defined in the conformal geometry, i.e. it does not depend on the choice of the Riemannian metric representing this conformal structure.

Theorem 1. *Let M^n be a smooth manifold equipped with two Riemannian metrics $g_{ij}(x)$, $\tilde{g}_{ij}(x)$ from the same conformal class:*

$$\tilde{g}_{ij} = e^{2\omega(x)} g_{ij}. \tag{1}$$

Let W^i_{jkl}, $\widetilde{W}^i_{jkl}$ denote the Weil tensors associated with the metrics $g_{ij}(x)$, $\tilde{g}_{ij}(x)$, respectively. Then

$$\widetilde{W}^i_{jkl} = W^i_{jkl}. \tag{2}$$

Proof. Let us simply calculate the transformation law for the Weil tensor

$$W_{ijkl} = R_{ijkl} - \left[g_{ik}P_{jl} + g_{jl}P_{ik} - g_{il}P_{jk} - g_{jk}P_{il}\right] \tag{3}$$

$$W^i{}_{jkl} = R^i{}_{jkl} - \left[\delta^i_k P_{jl} + g_{jl}g^{i\alpha}P_{\alpha k} - \delta^i_l P_{jk} - g_{jk}g^{i\alpha}P_{\alpha l}\right], \tag{4}$$

where P_{ij} is the Schouten tensor:

$$P_{ij} = \frac{1}{n-2}\left[R_{ij} - \frac{1}{2(n-1)} R\, g_{ij}\right].$$

Lemma 1. *Let the Riemannian metrics $g_{ij}(x)$, $\tilde{g}_{ij}(x)$ satisfy (1). Then*

1.

$$\begin{aligned}\widetilde{R}^i_{jkl} &= R^i_{jkl} + \delta^i_l\, \nabla_k \partial_j \omega - \delta^i_k\, \nabla_l \partial_j \omega + g_{kj}\, \nabla_l \partial^i \omega - g_{lj}\, \nabla_k \partial^i \omega + \\ &+ \delta^i_k\, \partial_l \omega\, \partial_j \omega - \delta^i_l\, \partial_k \omega\, \partial_j \omega + g_{lj}\, \partial^i \omega\, \partial_k \omega - g_{kj}\, \partial^i \omega\, \partial_l \omega + [\delta^i_l\, g_{kj} - \delta^i_k\, g_{lj}][\partial^\alpha \omega\, \partial_\alpha \omega].\end{aligned} \tag{5}$$

2.

$$\widetilde{P}_{ij} = P_{ij} - \nabla_i \partial_j \omega + \partial_i \omega \partial_j \omega - \frac{1}{2} g_{ij} \partial^\alpha \omega \partial_\alpha \omega \tag{6}$$

Remark 1. *During the calculation we shall use the following standard notations:*

$$\partial^k = g^{k\alpha}\partial_\alpha, \quad \nabla^k = g^{k\alpha}\nabla_\alpha,$$

the main property of the Levy-Civitta connection:

$$\nabla_k g_{ij} = 0, \quad \nabla_k g^{ij} = 0,$$

and the following relations:

$$\nabla_k \partial_j \omega = \nabla_j \partial_k \omega, \quad \nabla^k \partial_j \omega = \nabla_j \partial^k \omega.$$

Indeed

$$\nabla_k \partial_j \omega = \partial_k \partial_j \omega - \Gamma^{\alpha}_{kj} \partial_\alpha \omega = \partial_j \partial_k \omega - \Gamma^{\alpha}_{jk} \partial_\alpha \omega = \nabla_j \partial_k \omega,$$
$$\nabla^k \partial_j \omega = g^{k\alpha} \nabla_\alpha \partial_j \omega = g^{k\alpha} \nabla_j \partial_\alpha \omega = \nabla_j g^{k\alpha} \partial_\alpha \omega = \nabla_j \partial^k \omega.$$

Proof of formula (5). At the previous Lecture we proved:

$$\widetilde{R}^i_{jkl} = R^i_{jkl} + \nabla_k S^i_{lj} - \nabla_l S^i_{kj} + S^i_{k\alpha} S^\alpha_{lj} - S^i_{l\alpha} S^\alpha_{kj},$$

where

$$S^k_{ij} = \delta^k_i \, \partial_j \omega + \delta^k_j \, \partial_i \omega - g_{ij} \, \partial^k \omega.$$

Therefore

$$\nabla_k S^i_{lj} - \nabla_l S^i_{kj} = \nabla_k [\delta^i_l \, \partial_j \omega + \delta^i_j \, \partial_l \omega - g_{lj} \, \partial^i \omega] - \nabla_l [\delta^i_k \, \partial_j \omega + \delta^i_j \, \partial_k \omega - g_{kj} \, \partial^i \omega =$$
$$= \delta^i_l \, \nabla_k \partial_j \omega + \cancel{\delta^i_j \, \nabla_k \partial_l \omega} - g_{lj} \, \nabla_k \partial^i \omega - \delta^i_k \, \nabla_l \partial_j \omega - \cancel{\delta^i_j \, \nabla_l \partial_k \omega} + g_{kj} \, \nabla_l \partial^i \omega =$$
$$= \delta^i_l \, \nabla_k \partial_j \omega - \delta^i_k \, \nabla_l \partial_j \omega + g_{kj} \, \nabla_l \partial^i \omega - g_{lj} \, \nabla_k \partial^i \omega,$$

and

$$S^i_{k\alpha} S^\alpha_{lj} - S^i_{l\alpha} S^\alpha_{kj} =$$
$$= [\delta^i_k \, \partial_\alpha \omega + \delta^i_\alpha \, \partial_k \omega - g_{k\alpha} \, \partial^i \omega][\delta^\alpha_l \, \partial_j \omega + \delta^\alpha_j \, \partial_l \omega - g_{lj} \, \partial^\alpha \omega] -$$
$$-[\delta^i_l \, \partial_\alpha \omega + \delta^i_\alpha \, \partial_l \omega - g_{l\alpha} \, \partial^i \omega][\delta^\alpha_k \, \partial_j \omega + \delta^\alpha_j \, \partial_k \omega - g_{kj} \, \partial^\alpha \omega] =$$
$$= \delta^i_k \, \partial_l \omega \, \partial_j \omega + \cancel{\delta^i_k \, \partial_j \omega \, \partial_l \omega} - \delta^i_k \, g_{lj} \, \partial_\alpha \omega \, \partial^\alpha \omega +$$
$$+ \cancel{\delta^i_l \, \partial_k \omega \, \partial_j \omega} + \cancel{\delta^i_j \, \partial_k \omega \, \partial_l \omega} - \cancel{g_{lj} \, \partial_k \omega \, \partial^i \omega} -$$
$$- \cancel{g_{kl} \, \partial^i \omega \, \partial_j \omega} - g_{kj} \, \partial^i \omega \, \partial_l \omega + \cancel{g_{lj} \, \partial^i \omega \, \partial_k \omega} -$$
$$- \delta^i_l \, \partial_k \omega \, \partial_j \omega - \cancel{\delta^i_l \, \partial_j \omega \, \partial_k \omega} + \delta^i_l \, g_{kj} \, \partial_\alpha \omega \, \partial^\alpha \omega +$$
$$- \cancel{\delta^i_k \, \partial_l \omega \, \partial_j \omega} - \cancel{\delta^i_j \, \partial_l \omega \, \partial_k \omega} + \cancel{g_{kj} \, \partial_l \omega \, \partial^i \omega} -$$
$$+ \cancel{g_{lk} \, \partial^i \omega \, \partial_j \omega} + g_{lj} \, \partial^i \omega \, \partial_k \omega - \cancel{g_{kj} \, \partial^i \omega \, \partial_l \omega} =$$
$$= \delta^i_k \, \partial_l \omega \, \partial_j \omega - \delta^i_l \, \partial_k \omega \, \partial_j \omega + g_{lj} \, \partial^i \omega \, \partial_k \omega - g_{kj} \, \partial^i \omega \, \partial_l \omega + [\delta^i_l \, g_{kj} - \delta^i_k \, g_{lj}][\partial^\alpha \omega \, \partial_\alpha \omega].$$

Finally we obtain formula (5):

$$\widetilde{R}^i_{jkl} = R^i_{jkl} + \delta^i_l \, \nabla_k \partial_j \omega - \delta^i_k \, \nabla_l \partial_j \omega + g_{kj} \, \nabla_l \partial^i \omega - g_{lj} \, \nabla_k \partial^i \omega +$$
$$+ \delta^i_k \, \partial_l \omega \, \partial_j \omega - \delta^i_l \, \partial_k \omega \, \partial_j \omega + g_{lj} \, \partial^i \omega \, \partial_k \omega - g_{kj} \, \partial^i \omega \, \partial_l \omega + [\delta^i_l \, g_{kj} - \delta^i_k \, g_{lj}][\partial^\alpha \omega \, \partial_\alpha \omega].$$

Proof of formula (6). At the previous Lecture we proved:

$$\widetilde{R}_{jl} = R_{jl} - g_{jl} \nabla^\alpha \partial_\alpha \omega - (n-2) \nabla_l \partial_j \omega + (n-2) \partial_j \omega \partial_l \omega - (n-2) g_{jl} \partial^\alpha \omega \partial_\alpha \omega,$$

$$\widetilde{R} = e^{-2\omega}\big[R - 2(n-1)\nabla^\alpha\partial_\alpha\omega - (n-1)(n-2)\partial^\alpha\omega\partial_\alpha\omega\big],$$

therefore

$$\widetilde{P}_{ij} = P_{ij} + \frac{1}{n-2}\Bigg[-g_{ij}\nabla^\alpha\partial_\alpha\omega - (n-2)\nabla_i\partial_j\omega + (n-2)\partial_i\omega\partial_j\omega - (n-2)g_{ij}\partial^\alpha\omega\partial_\alpha\omega -$$

$$-\frac{g_{ij}}{2(n-1)}\big[-2(n-1)\nabla^\alpha\partial_\alpha\omega - (n-1)(n-2)\partial^\alpha\omega\partial_\alpha\omega\big]\Bigg] =$$

$$= P_{ij} + \frac{1}{n-2}\Bigg[-\cancel{g_{ij}\nabla^\alpha\partial_\alpha\omega} - (n-2)\nabla_i\partial_j\omega + (n-2)\partial_i\omega\partial_j\omega - (n-2)g_{ij}\partial^\alpha\omega\partial_\alpha\omega -$$

$$+\cancel{g_{ij}\nabla^\alpha\partial_\alpha\omega} + \frac{(n-2)}{2}g_{ij}\partial^\alpha\omega\partial_\alpha\omega\Bigg] =$$

$$= P_{ij} + \frac{1}{n-2}\Bigg[-(n-2)\nabla_i\partial_j\omega + (n-2)\partial_i\omega\partial_j\omega - \frac{(n-2)}{2}g_{ij}\partial^\alpha\omega\partial_\alpha\omega\Bigg] =$$

$$= P_{ij} - \nabla_i\partial_j\omega + \partial_i\omega\partial_j\omega - \frac{1}{2}g_{ij}\partial^\alpha\omega\partial_\alpha\omega.$$

□

Now we are ready to calculate the transformation law for the Weyl tensor

$$W^i{}_{jkl} = R^i{}_{jkl} - \big[\delta^i_k P_{jl} + g_{jl}g^{i\alpha}P_{\alpha k} - \delta^i_l P_{jk} - g_{jk}g^{i\alpha}P_{\alpha l}\big].$$

We have

$$\widetilde{R}^i_{jkl} = R^i_{jkl} + \delta^i_l\,\nabla_k\partial_j\omega - \delta^i_k\,\nabla_l\partial_j\omega + g_{kj}\,\nabla_l\partial^i\omega - g_{lj}\,\nabla_k\partial^i\omega +$$
$$+\,\delta^i_k\,\partial_l\omega\,\partial_j\omega - \delta^i_l\,\partial_k\omega\,\partial_j\omega + g_{lj}\,\partial^i\omega\,\partial_k\omega - g_{kj}\,\partial^i\omega\,\partial_l\omega + [\delta^i_l\,g_{kj} - \delta^i_k\,g_{lj}][\partial^\alpha\omega\,\partial_\alpha\omega],$$

$$\widetilde{P}_{ij} - \nabla_i\partial_j\omega + \partial_i\omega\partial_j\omega - \frac{1}{2}g_{ij}\partial^\alpha\omega\partial_\alpha\omega,$$

therefore

$$\tilde{W}^i{}_{jkl} = W^i{}_{jkl} + \delta^i_l\,\nabla_k\partial_j\omega - \delta^i_k\,\nabla_l\partial_j\omega + g_{jk}\,\nabla_l\partial^i\omega - g_{jl}\,\nabla_k\partial^i\omega +$$
$$+\,\delta^i_k\,\partial_j\omega\partial_l\omega - \delta^i_l\,\partial_j\omega\partial_k\omega + g_{jl}\,\partial^i\omega\partial_k\omega - g_{jk}\,\partial^i\omega\partial_l\omega + \big[\delta^i_l g_{jk} - \delta^i_k g_{jl}\big]\big[\partial^\alpha\omega\partial_\alpha\omega\big] -$$
$$-\,\delta^i_k\big[-\nabla_j\partial_l\omega + \partial_j\omega\partial_l\omega - \frac{1}{2}g_{jl}\partial^\alpha\omega\partial_\alpha\omega\big] + \delta^i_l\big[-\nabla_j\partial_k\omega + \partial_j\omega\partial_k\omega - \frac{1}{2}g_{jk}\partial^\alpha\omega\partial_\alpha\omega\big] -$$
$$-\,g_{jl}\big[-\nabla_k\partial^i\omega + \partial_k\omega\partial^i\omega - \frac{1}{2}\delta^i_k\partial^\beta\omega\partial_\beta\omega\big] + g_{jk}\big[-\nabla_l\partial^i\omega + \partial_l\omega\partial^i\omega - \frac{1}{2}\delta^i_l\partial^\beta\omega\partial_\beta\omega\big] = 0.$$

□

Remark. At the next lecture we will show that for $n = 3$ the Weyl tensor identically vanishes.