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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 8

Weyl and Cotton tensors

At this Lecture we consider the special case $n = 3$. Let us show that in this case:

1. The Weil tensor vanishes identically.
2. The Cotton tensor is invariant with respect to the conformal transformations of the metric tensor
$$\tilde{g}_{ij} = e^{2\omega(x)} g_{ij}, \tag{1}$$
therefore it is well defined in terms of conformal structure.

1) Let us prove the first statement. To do it, let us show, that

1. If $n = 2$, then the Riemann tensor at a given point is completely determined by the Riemann metric tensor and the scalar curvature calculated **at this point only**.
2. If $n = 3$, then the Riemann tensor at a given point is completely determined by the Riemann metric tensor and the Ricci tensor calculated **at this point only**.

Of course, if we know the metric tensor **in a neighborhood of this point**, it completely determines the Riemann tensor. But we discuss now only **algebraic** relations, not the **differential** ones.

Let us recall that the curvature tensor satisfy the following **algebraic** constraints:

1. Symmetry:
$$\begin{aligned} R_{ijlk} &= -R_{ijkl}, \\ R_{jikl} &= -R_{ijkl}, \\ R_{klij} &= R_{ijkl}. \end{aligned} \tag{2}$$

2. Biancki algebraic identities:
$$R_{sijk} + R_{sjki} + R_{skij} = 0.$$

It is easy to check that for $n = 2$ and $n = 3$ the Biancki algebraic identities follow from the symmetry properties (2) and provide no extra algebraic constraints, therefore we do not use them.

1. Let $n = 2$. The curvature tensor has 16 components, but only 4 of them do not vanish identically and all of them are proportional to R_{1212}:
$$\begin{aligned} R_{1221} &= -R_{1212}, \\ R_{2112} &= -R_{1212}, \\ R_{2121} &= R_{1212}. \end{aligned}$$

Let us calculate the scalar curvature omitting identically zero terms. We obtain:

$$R = g^{11}g^{22}R_{1212} + g^{22}g^{1}R_{2121} + g^{12}g^{21}R_{1221} + g^{21}g^{12}R_{2112} =$$
$$= 2(g^{11}g^{22} - g^{12}g^{21})R_{1212} = 2\det\left(g^{ij}\right)R_{1212} = \frac{2}{|g|}R_{1212},$$

where

$$|g| = \det(g_{ij}) = g_{11}g_{22} - g_{12}g_{21}.$$

Therefore

$$R_{1212} = \frac{|g|}{2}R,$$

and we expressed the Riemann curvature tensor through the metric tensor g_{ij} and the scalar curvature R **at this point.**

2. Let $n = 3$. Then

$$(g^{ij}) = \frac{1}{|g|}\begin{bmatrix} g_{22}g_{33} - g_{23}g_{32} & g_{13}g_{32} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ g_{23}g_{31} - g_{21}g_{33} & g_{11}g_{33} - g_{13}g_{31} & g_{21}g_{13} - g_{23}g_{11} \\ g_{32}g_{21} - g_{31}g_{22} & g_{31}g_{12} - g_{32}g_{11} & g_{11}g_{22} - g_{12}g_{21} \end{bmatrix}$$
$$(g_{ij}) = |g|\begin{bmatrix} g^{22}g^{33} - g^{23}g^{32} & g^{13}g^{32} - g^{12}g^{33} & g^{12}g^{23} - g^{13}g^{22} \\ g^{23}g^{31} - g^{21}g^{33} & g^{11}g^{33} - g^{13}g^{31} & g^{21}g^{13} - g^{23}g^{11} \\ g^{32}g^{21} - g^{31}g^{22} & g^{31}g^{12} - g^{32}g^{11} & g^{11}g^{22} - g^{12}g^{21} \end{bmatrix}$$

The Riemannian tensor has exactly 6 independent non-zero components:

$$R_{1212},\quad R_{1223},\quad R_{1231},\quad R_{2323},\quad R_{2331},\quad R_{3131}.$$

The Ricci tensor also has 6 non-zero components. Let us check that the Riemannian tensor can be express through the Ricci tensor.

For the scalar curvature tensor for $n = 3$ we have

$$R = R_{abcd}g^{ac}g^{bd} =$$
$$= R_{1212}g^{11}g^{22} + R_{1213}g^{11}g^{23} + R_{1221}g^{12}g^{21} + R_{1223}g^{12}g^{23} + R_{1231}g^{13}g^{21} + R_{1232}g^{13}g^{22} +$$
$$+ R_{1312}g^{11}g^{32} + R_{1313}g^{11}g^{33} + R_{1321}g^{12}g^{31} + R_{1323}g^{12}g^{33} + R_{1331}g^{13}g^{31} + R_{1332}g^{13}g^{32} +$$
$$+ R_{2112}g^{21}g^{12} + R_{2113}g^{21}g^{13} + R_{2121}g^{22}g^{11} + R_{2123}g^{22}g^{13} + R_{2131}g^{23}g^{11} + R_{2132}g^{23}g^{12} +$$
$$+ R_{2312}g^{21}g^{32} + R_{2313}g^{21}g^{33} + R_{2321}g^{22}g^{31} + R_{2323}g^{22}g^{33} + R_{2331}g^{23}g^{31} + R_{2332}g^{23}g^{32} +$$
$$+ R_{3112}g^{31}g^{12} + R_{3113}g^{31}g^{13} + R_{3121}g^{32}g^{11} + R_{3123}g^{32}g^{13} + R_{3131}g^{33}g^{11} + R_{3132}g^{33}g^{12} +$$
$$+ R_{3212}g^{31}g^{22} + R_{3213}g^{31}g^{23} + R_{3221}g^{32}g^{21} + R_{3223}g^{32}g^{23} + R_{3231}g^{33}g^{21} + R_{3232}g^{33}g^{22} =$$
$$= R_{1212}\left[2g^{11}g^{22} - 2g^{21}g^{12}\right] + R_{1313}\left[2g^{11}g^{33} - 2g^{31}g^{13}\right] + R_{2323}\left[2g^{22}g^{33} - g^{32}g^{23}\right] +$$
$$+ R_{1231}\left[4g^{13}g^{21} - 4g^{11}g^{23}\right] + R_{1223}\left[4g^{12}g^{23} - 4g^{13}g^{22}\right] + R_{3123}\left[4g^{32}g^{13} - 4g^{33}g^{12}\right].$$

Finally, by using symmetries, expressions for the determinants and collecting all the terms we obtain:

$$R = \frac{2}{|g|}\left[R_{1212}\,g_{33} + R_{2323}\,g_{11} + R_{3131}\,g_{22} + 2R_{1223}\,g_{13} + 2R_{1231}\,g_{23} + 2R_{3123}\,g_{12}\right].$$

Theorem 1. *Let $n = 3$. Then the Riemannian curvature tensor can be expressed through the Ricci tensor*

$$R_{abcd} = \big[g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} + g_{bc}R_{ad}\big] - \frac{R}{2}\big[g_{ac}g_{bd} - g_{bc}g_{ad}\big]. \tag{3}$$

Let us prove (3). Due to symmetry between the spatial variables it is sufficient to check 2 terms: R_{1212} and R_{1223}. Let us calculate the right-hand side of (3) for these terms.

a) Let $a = 1$, $b = 2$, $c = 1$, $d = 2$,

$$\begin{aligned}
&R_{1212} \overset{?}{=} \big[g_{11}R_{22} + g_{22}R_{11} - 2g_{12}R_{12}\big] - \frac{R}{2}\big[g_{11}g_{22} - g_{12}g_{12}\big] = \\
&= g_{11}\big[R_{1212}\,g^{11} + R_{1232}\,g^{13} + R_{3212}\,g^{31} + R_{3232}\,g^{33}\big] + \\
&+ g_{22}\big[R_{2121}\,g^{22} + R_{2131}\,g^{23} + R_{3121}\,g^{32} + R_{3131}\,g^{33}\big] - \\
&- 2g_{12}\big[R_{2112}\,g^{21} + R_{2132}\,g^{23} + R_{3112}\,g^{31} + R_{3132}\,g^{33}\big] - \\
&- \frac{R}{2}|g|g^{33} = \\
&= g_{11}\big[R_{1212}\,g^{11} - R_{1223}\,g^{13} - R_{1223}\,g^{31} + R_{2323}\,g^{33}\big] + \\
&+ g_{22}\big[R_{1212}\,g^{22} - R_{1231}\,g^{23} - R_{1231}\,g^{32} + R_{3131}\,g^{33}\big] + \\
&- 2g_{12}\big[- R_{1212}\,g^{21} + R_{2132}\,g^{23} + R_{1231}\,g^{31} - R_{3123}\,g^{33}\big] - \\
&- g^{33}\big[R_{1212}\,g_{33} + R_{2323}\,g_{11} + R_{3131}\,g_{22} + 2R_{1223}\,g_{13} + 2R_{1231}\,g_{23} + 2R_{3123}\,g_{12}\big] = \\
&= -R_{1212}\,\big[g^{11}g_{11} + g^{22}g_{22} + 2g^{12}g_{21} - g^{33}g_{33}\big] - R_{2323}\,\big[g^{33}g_{11} - g^{33}g_{11}\big] + \\
&+ R_{3131}\,\big[g^{33}g_{22} - g^{33}g_{22}\big] - 2R_{1223}\,\big[g^{31}g_{11} + g^{32}g_{21} + g^{33}g_{33} + g^{22}g_{22}\big] + \\
&- 2R_{1231}\,\big[g^{32}g_{22} + g^{31}g_{12} + g^{33}g_{32}\big] + 2R_{3123}\,\big[- g^{33}g_{12} - g^{33}g_{12}\big] = \\
&= R_{1212}\big[g^{11}g_{11} + g^{22}g_{22} + 2g^{12}g_{21} - g^{33}g_{33}\big].
\end{aligned}$$

Taking into account that

$$g^{11}g_{11} + g^{22}g_{22} + 2g^{12}g_{21} - g^{33}g_{33} =$$

$$= g^{11}g_{11} + g^{12}g_{21} + g^{13}g_{31} + g^{21}g_{21} + g^{22}g_{22} + g^{23}g_{32} - g^{31}g_{11} - g^{32}g_{23} - g^{33}g_{33} = 1 + 1 - 1 = 1,$$

we finish the proof.

b) Let $a=1,\ b=2,\ c=2,\ d=3$,

$$
\begin{aligned}
R_{1223} &\overset{?}{=} \big[g_{12}R_{23}+g_{23}R_{12}-g_{13}R_{22}+g_{22}R_{13}\big]-\frac{R}{2}\big[g_{12}g_{23}-g_{22}g_{13}\big]=\\
&=g_{12}\big[R_{1213}\,g^{11}+R_{1223}\,g^{12}+R_{3213}\,g^{31}+R_{3223}\,g^{32}\big]+\\
&+g_{23}\big[R_{2112}\,g^{21}+R_{2132}\,g^{23}+R_{3112}\,g^{31}+R_{3132}\,g^{33}\big]-\\
&-g_{13}\big[R_{1212}\,g^{11}+R_{1232}\,g^{13}+R_{3212}\,g^{31}+R_{3232}\,g^{33}\big]-\\
&-g_{22}\big[R_{2113}\,g^{21}+R_{2123}\,g^{22}+R_{3113}\,g^{31}+R_{3123}\,g^{32}\big]-\\
&-\frac{R}{2}|g|g^{13}=\\
&=g_{12}\big[-R_{1231}\,g^{11}+R_{1223}\,g^{12}+R_{3123}\,g^{31}-R_{2323}\,g^{32}\big]+\\
&+g_{23}\big[-R_{1212}\,g^{21}+R_{2132}\,g^{23}+R_{1231}\,g^{31}-R_{3123}\,g^{33}\big]-\\
&-g_{13}\big[R_{1212}\,g^{11}-R_{1223}\,g^{13}-R_{1223}\,g^{31}+R_{2323}\,g^{33}\big]-\\
&-g_{22}\big[R_{1231}\,g^{21}-R_{1223}\,g^{22}-R_{3131}\,g^{31}+R_{3123}\,g^{32}\big]-\\
&-g^{13}\big[R_{1212}\,g_{33}+R_{2323}\,g_{11}+R_{3131}\,g_{22}+2R_{1223}\,g_{13}+2R_{1231}\,g_{23}+2R_{3123}\,g_{12}\big]=\\
&=-R_{1212}\,\big[g^{12}g_{23}+g^{11}g_{13}+g^{13}g_{33}\big]-R_{2323}\,\big[g^{32}g_{21}+g^{33}g_{31}+g^{31}g_{11}\big]+\\
&+R_{3131}\,\big[g^{31}g_{22}-g^{31}g_{22}\big]+R_{1223}\,\big[g^{21}g_{12}+g^{23}g_{32}+2g^{13}g_{13}+g^{22}g_{22}-2g^{13}g_{13}\big]+\\
&-R_{1231}\,\big[g^{11}g_{12}-g^{13}g_{32}+g^{12}g_{22}+2g^{13}g_{32}\big]-\\
&-R_{3123}\,\big[-g^{31}g_{12}+g^{33}g_{32}+g^{32}g_{22}+2g^{31}g_{12}\big]=R_{1223}.
\end{aligned}
$$

Corollary 1. *Let $n=3$. Then the Weil tensor vanish identically $W^i_{jkl}\equiv 0$.*

Proof. For $n=3$ the Weil tensor with lower indexes W_{ijkl} coincides with the difference between the right-hand side of (3) and the left hand side of (3). Therefore for $n=3$ $W_{ijkl}\equiv 0$. □

2) Consider the following transformation of the Riemannian metric preserving the conformal class:

$$\tilde g_{ij}=e^{2\omega(x)}g_{ij}, \tag{4}$$

Let us calculate the transformation law for the Cotton tensor.

Remark 1. *During the calculation we shall use the following standard notations:*

$$\partial^k=g^{k\alpha}\partial_\alpha,\quad \nabla^k=g^{k\alpha}\nabla_\alpha,$$

the main property of the Levy-Civitta connection:

$$\nabla_k g_{ij}=0,\quad \nabla_k g^{ij}=0,$$

and the following relations:

$$\nabla_k\partial_j\omega=\nabla_j\partial_k\omega,\quad \nabla^k\partial_j\omega=\nabla_j\partial^k\omega,$$

$$(\nabla_k \nabla_l - \nabla_l \nabla_k)\alpha_q = -R^p{}_{qkl}\alpha_p, \tag{5}$$

where α_p is an 1-form.

Let us recall the formulas for the Schouten tensor

$$P_{ij} = \frac{1}{n-2}\left[R_{ij} - \frac{1}{2(n-1)}R\, g_{ij}\right]$$

and the Cotton tensor

$$C_{ijk} = (n-2)\left[\nabla_k P_{ij} - \nabla_j P_{ik}\right].$$

Consider the following change of metric:

Theorem 2. *The change of Riemannian metric tensor (4) generates the following transformation of the Cotton tensor for any $n \geq 3$:*

$$\tilde{C}_{jkl} = C_{jkl} - (n-2)\partial_i \omega W^i{}_{jkl}.$$

We proved today that for $n = 3$ the Weil tensor vanishes identically $W^i_{jkl} \equiv 0$.. Therefore we have:

Corollary 2. *If $n = 3$ then the Cotton tensor is invariant with respect to the transformation (4), and it provides an invariant of conformal geometry.*

Proof. We have the following transformation laws for the Levi-Civita connection

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + S^k_{ij},$$

where

$$S^k_{ij} = \delta^k_i\, \partial_j \omega + \delta^k_j\, \partial_i \omega - g_{ij}\, \partial^k \omega,$$

and for the Schouten tensor

$$\widetilde{P}_{ij} = P_{ij} - \nabla_i \partial_j \omega + \partial_i \omega \partial_j \omega - \frac{1}{2} g_{ij} \partial^\alpha \omega \partial_\alpha \omega,$$

therefore

$$\frac{1}{n-2}\tilde{C}_{jkl} = \tilde{\nabla}_l \tilde{P}_{jk} - \tilde{\nabla}_k \tilde{P}_{jl} = \nabla_l \tilde{P}_{jk} - S^\alpha_{lj}\tilde{P}_{\alpha k} - \cancel{S^\alpha_{lk}\tilde{P}_{j\alpha}} - \nabla_k \tilde{P}_{jl} + S^\alpha_{kj}\tilde{P}_{\alpha l} + \cancel{S^\alpha_{kl}\tilde{P}_{j\alpha}} =$$

$$= \nabla_l \left[P_{jk} - \nabla_k \partial_j \omega + \partial_j \omega \partial_k \omega - \frac{1}{2} g_{jk} \partial^\beta \omega \partial_\beta \omega\right] -$$

$$- \nabla_k \left[P_{jl} - \nabla_l \partial_j \omega + \partial_j \omega \partial_l \omega - \frac{1}{2} g_{jl} \partial^\beta \omega \partial_\beta \omega\right] -$$

$$- \left[\delta^\alpha_l\, \partial_j \omega + \delta^\alpha_j\, \partial_l \omega - g_{lj}\, \partial^\alpha \omega\right] \cdot \left[P_{\alpha k} - \nabla_k \partial_\alpha \omega + \partial_\alpha \omega \partial_k \omega - \frac{1}{2} g_{\alpha k} \partial^\beta \omega \partial_\beta \omega\right] +$$

$$+ \left[\delta^\alpha_k\, \partial_j \omega + \delta^\alpha_j\, \partial_k \omega - g_{kj}\, \partial^\alpha \omega\right] \cdot \left[P_{\alpha l} - \nabla_l \partial_\alpha \omega + \partial_\alpha \omega \partial_l \omega - \frac{1}{2} g_{\alpha l} \partial^\beta \omega \partial_\beta \omega\right] =$$

$$= \left[\nabla_l P_{jk} - \nabla_k P_{jl}\right] + \left[-\nabla_l \nabla_k \partial_j \omega + \nabla_k \nabla_l \partial_j \omega\right] -$$

$$- \left[\delta^\alpha_l\, \partial_j \omega + \delta^\alpha_j\, \partial_l \omega - g_{lj}\, \partial^\alpha \omega\right] P_{\alpha k} + \left[\delta^\alpha_k\, \partial_j \omega + \delta^\alpha_j\, \partial_k \omega - g_{kj}\, \partial^\alpha \omega\right] P_{\alpha l} + \mathcal{C}_2 + \mathcal{C}_3,$$

where

$$\mathcal{C}_2 = \nabla_l \left[\partial_j\omega\partial_k\omega - \frac{1}{2}g_{jk}\partial^\alpha\omega\partial_\alpha\omega\right] - \nabla_k \left[\partial_j\omega\partial_l\omega - \frac{1}{2}g_{jl}\partial^\alpha\omega\partial_\alpha\omega\right] -$$
$$+ \left[\delta_l^\alpha\,\partial_j\omega + \delta_j^\alpha\,\partial_l\omega - g_{lj}\,\partial^\alpha\omega\right]\nabla_k\partial_\alpha\omega - \left[\delta_k^\alpha\,\partial_j\omega + \delta_j^\alpha\,\partial_k\omega - g_{kj}\,\partial^\alpha\omega\right]\nabla_l\partial_\alpha\omega =$$
$$= \cancel{\partial_j\omega\nabla_l\partial_k\omega} + \cancel{\partial_k\omega\nabla_l\partial_j\omega} - \cancel{g_{jk}\partial^\alpha\omega\nabla_l\partial_\alpha\omega} - \cancel{\partial_j\omega\nabla_k\partial_l\omega} - \cancel{\partial_l\omega\nabla_k\partial_j\omega} + \cancel{g_{jl}\partial^\alpha\omega\nabla_k\partial_\alpha\omega} +$$
$$+ \cancel{\partial_j\omega\nabla_k\partial_l\omega} + \cancel{\partial_l\omega\nabla_k\partial_j\omega} - \cancel{g_{lj}\partial^\alpha\omega\nabla_k\partial_\alpha\omega} - \cancel{\partial_j\omega\nabla_l\partial_k\omega} - \cancel{\partial_k\omega\nabla_l\partial_j\omega} + \cancel{g_{kj}\partial^\alpha\omega\nabla_l\partial_\alpha\omega} = 0,$$

$$\mathcal{C}_3 = -\left[\delta_l^\alpha\,\partial_j\omega + \delta_j^\alpha\,\partial_l\omega - g_{lj}\,\partial^\alpha\omega\right]\cdot\left[\partial_\alpha\omega\partial_k\omega - \frac{1}{2}g_{\alpha k}\partial^\beta\omega\partial_\beta\omega\right] +$$
$$+ \left[\delta_k^\alpha\,\partial_j\omega + \delta_j^\alpha\,\partial_k\omega - g_{kj}\,\partial^\alpha\omega\right]\cdot\left[\partial_\alpha\omega\partial_l\omega - \frac{1}{2}g_{\alpha l}\partial^\beta\omega\partial_\beta\omega\right] =$$
$$= -\cancel{\partial_j\omega\partial_l\omega\partial_k\omega} - \cancel{\partial_l\omega\partial_j\omega\partial_k\omega} + \underline{g_{lj}\partial_k\omega\partial^\alpha\omega\partial_\alpha\omega} +$$
$$+ \cancel{\frac{1}{2}g_{kl}\partial_j\omega\partial^\beta\omega\partial_\beta\omega} + \underline{\underline{\frac{1}{2}g_{jk}\partial_l\omega\partial^\beta\omega\partial_\beta\omega}} - \underline{\frac{1}{2}g_{jl}\partial_k\omega\partial^\beta\omega\partial_\beta\omega} +$$
$$+ \cancel{\partial_j\omega\partial_k\omega\partial_l\omega} + \cancel{\partial_k\omega\partial_j\omega\partial_l\omega} - \underline{\underline{g_{kj}\partial_l\omega\partial^\alpha\omega\partial_\alpha\omega}} +$$
$$- \cancel{\frac{1}{2}g_{kl}\partial_j\omega\partial^\beta\omega\partial_\beta\omega} - \underline{\frac{1}{2}g_{jl}\partial_k\omega\partial^\beta\omega\partial_\beta\omega} + \underline{\underline{\frac{1}{2}g_{jk}\partial_l\omega\partial^\beta\omega\partial_\beta\omega}} = 0.$$

Therefore,

$$\frac{1}{n-2}\tilde{C}_{jkl} = \frac{1}{n-2}C_{jkl} - R^i{}_{jkl}\partial_j\omega - \cancel{\partial_j\omega P_{lk}} - \partial_l\omega P_{jk} + g_{lj}\,\partial^\alpha\omega P_{\alpha k} +$$
$$+ \cancel{\partial_j\omega P_{kl}} + \partial_k\omega P_{jl} - g_{kj}\,\partial^\alpha\omega P_{\alpha l}$$

But

$$W^i{}_{jkl} = R^i{}_{jkl} - \left[\delta_k^i P_{jl} + g_{jl}g^{i\alpha}P_{\alpha k} - \delta_l^i P_{jk} - g_{jk}g^{i\alpha}P_{\alpha l}\right],$$

and

$$\partial_i\omega W^i{}_{jkl} = \partial_i\omega R^i{}_{jkl} - \partial_k\omega P_{jl} - g_{jl}\,\partial^\alpha\omega P_{\alpha k} + \partial_l\omega P_{jk} + g_{jk}\,\partial^\alpha\omega P_{\alpha l},$$

what completes the proof. □