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Lecture notes

Conformal geometry and Riemann surfaces

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Lecture 8

Beltrami equation

Starting from this Lecture we will concentrate on the case $n = 2$.

The aim of the next two Lectures is to prove that all 2-dimensional conformal Riemannian manifold are **conformally flat.**

More precisely:

Theorem 1. *Let $(M^2, g_{ij}(\vec{x}))$ be a 2-dimensional Riemannian manifold, $\vec{x}_0$ be a point of M^2. Then there exist a neighborhood $U(\vec{x}_0))$ and a local coordinate system (u, v) at U such that in the coordinates (u, v) the metric tensor has the form*

$$g_{ij}(u, v) = e^{2\omega(u,v)}\delta_{ij}, \tag{1}$$

where $\omega(u, v)$ is scalar real function.

Definition 1. *Coordinates u, v such that $g_{ij}(u, v) = e^{2\omega(u,v)}\delta_{ij}$ are called **isothermal.***

How to construct them?

Definition 2. *Let M^2 be a smooth **oriented** 2-dimensional manifold equipped with a Riemannian metric*

$$G = \begin{pmatrix} g_{11} & g_{12}, \\ g_{21} & g_{22} \end{pmatrix}.$$

*The **quasicomplex structure** on M^2 is a linear operator J acting on the tangent bundle TM^2 such that*

1. *For any point $\vec{x}_0 \in M^2$ it maps the tangent space to this point $T_{\vec{x}_0}M^2$ onto itself.*
2. *The restriction of J to the space $T_{\vec{x}_0}M^2$ is isometry with respect to the Riemannian metric g_{jk}; moreover, it is the 90 degrees counterclockwise rotation.*
3. *$J^2 = -1$.*

Proposition 1. *Let (x^1, x^2) be a **positive** coordinate system on M^2. Then the we have the following formula for J:*

$$J = \frac{1}{\sqrt{|G|}} \begin{pmatrix} -g_{12} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix}.$$

Proof. Consider the tangent space to a fixed point $\vec{x}_0$. Let:

$$J = \begin{pmatrix} a_{11} & a_{12}, \\ a_{21} & a_{22} \end{pmatrix}.$$

Since $Je_1 = \left(\begin{array}{c} a_{11} \\ a_{21} \end{array} \right)$, $Je_2 = \left(\begin{array}{c} a_{12} \\ a_{22} \end{array} \right)$, we can explicitly write the conditions on J:

$$[a_{11}a_{21}] \begin{pmatrix} g_{11} & g_{12}, \\ g_{21} & g_{22} \end{pmatrix} \left(\begin{array}{c} 1 \\ 0 \end{array} \right) = 0, \quad [a_{12}a_{22}] \begin{pmatrix} g_{11} & g_{12}, \\ g_{21} & g_{22} \end{pmatrix} \left(\begin{array}{c} 0 \\ 1 \end{array} \right) = 0$$

Hence, $a_{11}g_{11} + a_{21}g_{12} = 0$ and $\left(\begin{array}{c} a_{11} \\ a_{21} \end{array} \right) = C_1 \left(\begin{array}{c} -g_{12} \\ g_{11} \end{array} \right)$. Since we want a rotation in a positive direction, $C_1 > 0$. Similarly, $\left(\begin{array}{c} a_{12} \\ a_{22} \end{array} \right) = C_2 \left(\begin{array}{c} -g_{22} \\ g_{12} \end{array} \right)$, $C_2 > 0$. Then we have

$$C_1^2[-g_{12} \; g_{11}] \begin{pmatrix} g_{11} & g_{12}, \\ g_{21} & g_{22} \end{pmatrix} \left(\begin{array}{c} -g_{12} \\ g_{11} \end{array} \right) = g_{11}$$

Then

$$C_1^2[0 \; g_{11}g_{22} - g_{12}^2] \left(\begin{array}{c} -g_{12} \\ g_{11} \end{array} \right) = g_{11}.$$

Therefore,

$$C_1 = \frac{1}{g_{11}|G|}.$$

Finally,

$$Je_1 = \frac{1}{\sqrt{|G|}} \left(\begin{array}{c} -g_{12} \\ g_{11} \end{array} \right),$$

$$Je_2 = \frac{1}{\sqrt{|G|}} \left(\begin{array}{c} -g_{22} \\ g_{12} \end{array} \right).$$

Hence, we proved that

$$J = \frac{1}{\sqrt{|G|}} \begin{pmatrix} -g_{12} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix}.$$

$\square$

One can easily check that

$$J^2 = \frac{1}{|G|} \begin{pmatrix} g_{12}^2 - g_{11}g_{22} & 0 \\ 0 & g_{12}^2 - g_{11}g_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we ask ourselves: how to find the isothermic coordinates? If coordinates (u, v) are isothermal, then

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us introduce complex coordinate: $w = u + iv$. We already know $\partial_{\bar{z}} z = 0$. One can write

$$\partial_{\overline{w}} = \frac{1}{2} \left[\partial_u + i\partial_v \right] = \frac{1}{2} \left[\partial_u + iJ\partial_u \right].$$

It is easy to check that

Lemma 1. *Let a vector field $\vec{W} \neq 0$ in a neighborhood of a point $\vec{x}_0$. Then a local coordinate system (u, v) is isothermal iff the function $w = u + iv$ satisfies Beltrami equation*

$$[L_{\vec{W}} + L_{J\vec{W}}]\, w = 0, \tag{2}$$

where $L_{\vec{W}}$ denotes the directional derivative along the vector field $\vec{W}$.

Therefore, to construct local isothermal coordinates near a point $\vec{x}_0$, it is sufficient to construct local solution of the Beltrami equation such that $L_{\vec{W}} w \neq 0$ at the point $\vec{x}_0$ and to define coordinates (u, v) by

$$u = \text{Re } w, \quad v = \text{Im } w.$$

Our next step is to prove existence of Beltrami equation solutions!

Beltrami equation

Let (x, y) be local coordinates near the point $\vec{x}_0$. We can choose $\vec{W} = \partial_x$, and the Beltrami equation takes the form

$$\overline{\mathcal{D}} w = 0, \quad \text{where } \overline{\mathcal{D}} = \partial_x + iJ\partial_x. \tag{3}$$

Using Propositions 1 we obtain

$$\overline{\mathcal{D}} = \partial_x + \frac{i}{|G|}[-g_{12}\partial_x + g_{11}\partial_y].$$

Using that

$$\partial_x = \partial_z + \partial_{\overline{z}}, \quad \partial_y = i[\partial_z - \partial_{\overline{z}}]$$

we obtain

$$\overline{\mathcal{D}} = \partial_z + \partial_{\overline{z}} - \frac{ig_{12}}{\sqrt{|G|}}[\partial_z + \partial_{\overline{z}}] + \frac{i}{\sqrt{|G|}} ig_{11}[\partial_z - \partial_{\overline{z}}] = a(z, \overline{z})\partial_{\overline{z}} + b(z, \overline{z})\partial_z,$$

where

$$a(z, \overline{z}) = \left[1 + \frac{g_{11}}{\sqrt{|G|}} - \frac{ig_{12}}{\sqrt{|G|}}\right], \quad b(z, \overline{z}) = \left[1 - \frac{g_{11}}{\sqrt{|G|}} - \frac{ig_{12}}{\sqrt{|G|}}\right].$$

$g_{11} > 0$, therefore

$$|\text{Re } a(z, \overline{z})| > |\text{Re } b(z, \overline{z})|, \quad |\text{Im } a(z, \overline{z})| = |\text{Im } b(z, \overline{z})|,$$

and

$$|\alpha(z, \overline{z})| < 1, \quad \text{where } \alpha(z, \overline{z}) = \frac{a(z, \overline{z})}{b(z, \overline{z})}.$$

$|a| > 0$, therefore Equation 3 is equivalent to

$$\frac{1}{a} \overline{\mathcal{D}}\, w = 0,$$

and finally we transform Beltrami equation to the following form

$$[\partial_{\overline{z}} + \alpha(z,\overline{z})\partial_z]w(z,\overline{z}) = 0. \tag{4}$$

We are looking for **local** solutions of (4), therefore without loss of generality we may assume that α has compact support.

We are looking for non-zero solutions of this equation. It is convenient to replace this differential equation by an integral one. Let us try to find a solution of (4) in the following form:

$$w(z,\overline{z}) = \phi(z) + \partial_{\overline{z}}^{-1} f(z,\overline{z}), \tag{5}$$

where $\phi(z)$ is a holomorphic function. Equation 4 is equivalent to

$$f(z,\overline{z}) = -\alpha(z,\overline{z})\phi'(z) - \alpha(z,\overline{z})\,\partial_z\partial_{\overline{z}}^{-1}\, f(z,\overline{z}). \tag{6}$$

It is natural to solve (7) using the standard iteration procedure

$$f_0(z,\overline{z}) = -\alpha(z,\overline{z})\phi'(z), \quad f_{k+1}(z,\overline{z}) = -\alpha(z,\overline{z})\phi'(z) - \alpha(z,\overline{z})\,\partial_z\partial_{\overline{z}}^{-1}\, f_k(z,\overline{z}). \tag{7}$$

It is important to choose a proper functional space. We shall use the Sobolev space $H = W^{l,2}(\mathbb{R}^2)$ with a "weighted" norm, depending on an extra real parameter $\lambda > 0$. Denote by $\hat{f}(p,q)$ the Fourier transform of $f(x,y)$:

$$f(x,y) = \frac{1}{2\pi}\int\int e^{ipx+iqy}\hat{f}(p,q)dpdq,$$

The scalar product $<\,,\,>$ in the space $W^{l,2}(\mathbb{R}^2)$ is defined by

$$\begin{aligned}
< f,g > &= \int\int \overline{\hat{f}(p,q)}\hat{g}(p,q)(1+\lambda^2(p^2+q^2))^l dpdq = \\
&= \int\int \left[\sum_{k=0}^{l}\sum_{j=0}^{l=k}\binom{l}{j+k}\binom{j+k}{j}\lambda^{2j+2k}p^{2j}q^{2k}\right]\overline{\hat{f}(p,q)}\hat{g}(p,q)dpdq = \\
&= \sum_{k=0}^{l}\sum_{j=0}^{l=k}\binom{l}{j+k}\binom{j+k}{j}\lambda^{2j+2k}\left(\partial_x^j\partial_y^k f, \partial_x^j\partial_y^k g\right),
\end{aligned} \tag{8}$$

where $(\ ,\ )$ denotes the standard L^2 scalar product

$$(f,g) = \int\int \overline{f(x,y)}g(x,y)dxdy = \int\int \overline{\hat{f}(p,q)}\hat{g}(p,q)dpdq. \tag{9}$$

We see that

$$\left|\left(\partial_x^j\partial_y^k f, \partial_x^j\partial_y^k g\right)\right| \leq \frac{1}{\lambda^{2j+2k}} < f,g > . \tag{10}$$

The Fourier transform maps the partial derivatives to multiplication operators:

$$\partial_x \to ip,\ \partial_y \to iq,\ \partial_z \to \frac{i}{2}(p+iq).$$

Hence,

$$\partial_z^{-1}\partial_{\overline{z}} \to \left(\frac{p-iq}{p+ig}\right),$$

therefore $\partial_z^{-1}\partial_{\overline{z}}$ is a unitary operator in H. It is easy to check

Lemma 2. *If $\alpha(x,y)$ is a smooth function with a compact support and $|\alpha(x,y)| \leq C < 1$, then for sufficiently small λ norm of multiplication by $\alpha(x,y)$ in the space H is smaller then* 1.

Therefore for sufficiently small λ the operator $\alpha(z,\overline{z})\,\partial_z \partial_{\overline{z}}^{-1}$ is contracting and Equation 7 has an unique solution for arbitrary $\phi(z)$.

Let us recall the proof of the Sobolev's embedding theorem.

Assume that α is k times differentiable, $f \in W^{l,2}(\mathbb{R}^2)$, which means

$$\int\int (1+p^2+q^2)^l |f(p,q)| dpdq < \infty.$$

By Cauchy Bunyakovski ineqaulity,

$$\left[\int |fg|\right]^2 \leqslant \int |f|^2 \int |g|^2.$$

By taking $f(p,q) = (1+p^2+q^2)^{-1}$ and $g = (1+p^2+q^2)^{l/2} f(p,q)$ and putting it into the inequality, we have

$$\int |(1+p^2+q^2))^{\frac{k-2}{2}} f(p,q)| dpdq < \infty.$$

We proved that when $l=2$, $\hat{f}(p,q) \in L^1$ and f is continuous; when $l=3$, $f_z,\ f_{\overline{z}} \in L^1$ and f is differentiable. Therefore, the isothermic coordinate exists at least locally and we have local flatness.