Problems

1. Calculate the Riemannian metric for the Poincaré model of Lobachevsky plane.

2. Calculate the metric on S^2 in coordinates generated by the stereographic projection. Check that the stereographic projection is a conformal map from S^2 to R^2.

3. Construct a fractional-linear map
$$z \to \frac{az+v}{cz+d},$$
from the open unit disk $|z| < 1$ to the upper half-plane $\operatorname{Im} z > 0$.

4. Calculate the Riemannian metric for the upper half-plane model of Lobachevsky plane.

5. Let z_1, z_2, z_3, z_4 be points in the complex plane. Find the necessary and sufficient condition for these four points to lie on the same circle.

6. Calculate the subgroup of $SL(2,\mathbb{C})$ of the transformation leaving invariant the open unit disk $|z| < 1$.

7. Show that the transformations leaving invariant the open unit disk $|z| < 1$ are isometries of the unit disk wit the metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2}.$$

8. Show that the subgroup of $SL(2,\mathbb{C})$ of the transformation leaving invariant the upper half-plane $\operatorname{Im} z > 0$ coincides with $SL(2,\mathbb{R})$.

9. Show that the transformations leaving invariant the upper half-plane $\operatorname{Im} z > 0$ are isometries of the upper half-plane with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

10. Calculate the curvature tensor for the Lobachevsky plane.

11. Let z_1, z_2 be a pair of points in the upper half-plane. Calculate the Lobachevsky distance between these points.

12. Consider a triangle in the 2-dimensional sphere S^2. Assume that the radius of this sphere is equal to 1. Denote the lengths of the sides by A, B, C. Denote the angles opposite to these sides by α, β, γ respectively.

    Proof the spherical cosine theorem:

$$\begin{aligned}\cos A &= \cos B \cos C + \sin B \sin C \cos \alpha,\\ \cos \alpha &= -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos A.\end{aligned}$$

13. Consider a triangle in the 2-dimensional sphere S^2. Assume that the radius of this sphere is equal to 1. Denote the lengths of the sides by A, B, C. Denote the angles opposite to these sides by α, β, γ respectively. Proof the spherical sine theorem:

$$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B} = \frac{\sin \gamma}{\sin C}.$$

14. Consider a triangle in the Lobachevsky plane. Denote the lengths of the sides by A, B, C. Denote the angles opposite to these sides by α, β, γ respectively. Proof the cosine theorem:

$$\begin{aligned}\operatorname{ch} A &= \operatorname{ch} B \operatorname{ch} C - \operatorname{sh} B \operatorname{sh} C \cos \alpha,\\ \cos \alpha &= -\cos \beta \cos \gamma + \sin \beta \sin \gamma \operatorname{ch} A.\end{aligned}$$

15. Consider a triangle in the Lobachevsky plane. Denote the lengths of the sides by A, B, C. Denote the angles opposite to these sides by α, β, γ respectively. Proof the sine theorem:

$$\frac{\sin\alpha}{\operatorname{sh} A} = \frac{\sin\beta}{\operatorname{sh} B} = \frac{\sin\gamma}{\operatorname{sh} C}.$$

16. Calculate the length of a circle of radius R in the Lobachevsky plane.

17. Calculate the ares of a disk of radius R in the Lobachevsky plane.

18. Define the vector product on the vectors in the pseudoeuclidean space $R^{2,1}$.

19. Find the basis of holomorphic 2-forms for the surfaces

    - $\mu^2 = \prod_{k=1}^{3} (\lambda - \lambda_k),$
    - $\mu^2 = \prod_{k=1}^{5} (\lambda - \lambda_k),$
    - $\mu^2 = \prod_{k=1}^{7} (\lambda - \lambda_k).$