

# An exact solution for spin and charge correlations in quantum dots

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THE SCIENCE OF NANOSTRUCTURES: New Frontiers in the Physics of Quantum Dots

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In collaboration with

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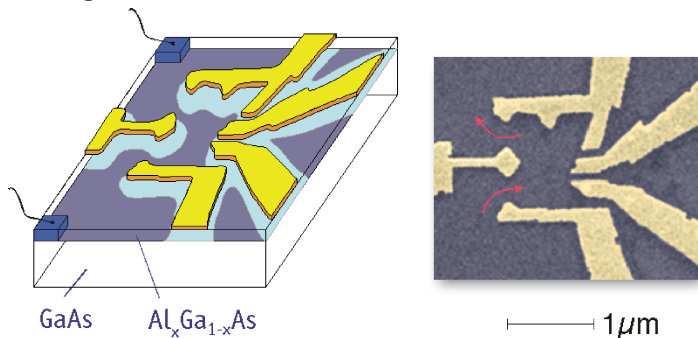
**Mikhail Kiselev** (ICTP, Italy)

**Azat Sharafutdinov** (Landau Institute, Russia)

An **approximate** solution

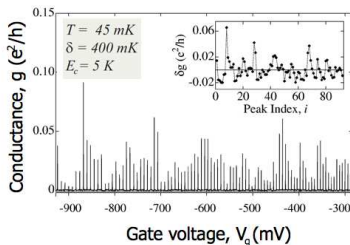
next talk by Alexander Shnirman

- ▶ Transport through quantum dots (QDs)
  - ▶ metallic grains, e.g. Al, Au, Pd, with diameter 1 – 10 nm
  - ▶ large molecules
  - ▶ regions of confinement for electrons in 2DEG with size  $\sim 1 \mu m$

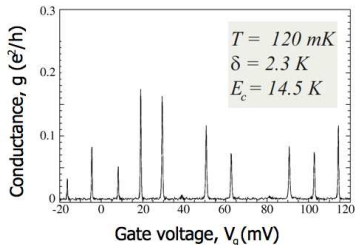


[adopted from Kouwenhoven, Marcus (1998)]

from Patel et al. (1998)



from Luscher et al. (2001)

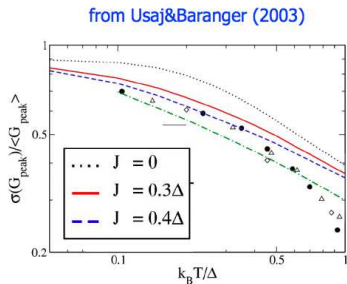
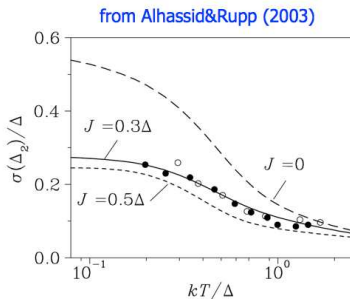


No periodicity in gate voltage

Different spacings between conductance peaks

Different heights of conductance peaks

Possible explanation: the effect of single-particle level fluctuations

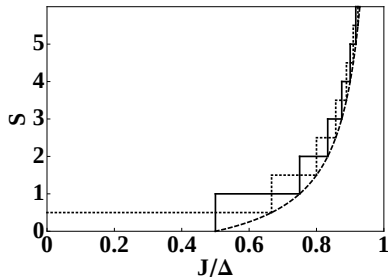
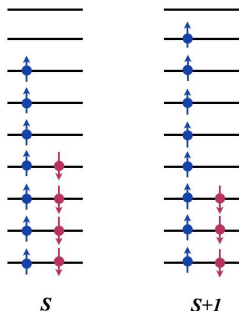


Data points from Patel et al.

Theory is applicable at relatively low temperatures:

Exchange  $J > 0$  is important

hamiltonian with ferromagnetic exchange  $H_J = -J\hat{S}^2$



ground state energies for spin  $S$  and  $S + 1$

$$E_{S+1} - E_S = (2S + 1)\delta - J(2S + 2) \implies S = \frac{2J - \delta}{2(\delta - J)}$$

renormalized exchange  $J_\star = 2JS = \frac{J\delta}{\delta - J} \gg J$  at  $\delta - J \ll \delta$

[Kurland, Aleiner, Altshuler (2000)]

- ▶ can we find signatures of **exchange  $J$**  in physical observables, e.g. conductance, tunneling DOS, spin susceptibility, at  $T \gg \delta, J$ ?
- ▶ how the single-particle **level statistics** affects results, in particular, the mesoscopic Stoner instability, at  $T \gg \delta, J$ ?

$$\mathcal{H} = \sum_{\alpha,\sigma} \epsilon_{\alpha,\sigma} a_{\alpha,\sigma}^\dagger a_{\alpha,\sigma} + E_c \left( \hat{N} - N_0 \right)^2 - J \hat{\mathbf{S}}^2$$

$$\epsilon_{\alpha,\sigma} = \epsilon_\alpha + b\sigma/2 \quad - \quad \begin{array}{l} \text{(random) single particle spectrum} \\ \text{with Zeeman splitting } b = g\mu_B B \end{array}$$

$$\hat{N} = \sum_{\alpha,\sigma} a_{\alpha,\sigma}^\dagger a_{\alpha,\sigma} \quad - \quad \text{particle number operator}$$

$$\hat{\mathbf{S}}_{\sigma\sigma'} = \frac{1}{2} \sum_{\alpha,\sigma} a_{\alpha,\sigma}^\dagger \boldsymbol{\sigma}_{\sigma\sigma'} a_{\alpha,\sigma'} \quad - \quad \text{spin operator}$$

[Kurland, Aleiner, Altshuler (2000); Aleiner, Brouwer, Glazman (2002)]

assumptions:

good metal:  $E_{\text{Th}}/\delta \gg 1$ , and low temperatures:  $T \ll E_{\text{Th}}, E_c$

targets:

static spin susceptibilities  $\chi = T \frac{\partial^2}{\partial b^2} \ln \text{Tr} \exp(-\beta \mathcal{H})$

tunneling DOS:

$$\nu(\varepsilon) = -\frac{1}{\pi} \text{Im} \sum_{\alpha,\sigma} \mathcal{G}_{\alpha\sigma;\alpha\sigma}^R(\varepsilon), \quad \mathcal{G}^R(t_1, t_2) = -i\theta(t_1 - t_2) \langle \{ a(t_1), a^\dagger(t_2) \} \rangle$$

- Hubbard-Stratonovich transformation of exchange interaction:

$$e^{\mp i t J \hat{\mathbf{S}}^2} = \lim_{N \rightarrow \infty} \prod_{\alpha} \prod_{n=1}^N \int d\theta_n e^{\pm \frac{i}{4J} t \theta_n^2 / N} e^{i t \theta_n \hat{\mathbf{S}}_{\alpha} / N} = \prod_{\alpha} \int \mathcal{D}[\theta] e^{\pm \frac{i}{4J} \int_0^t dt' \theta^2} \mathcal{T} e^{i \int_0^t dt' \theta \hat{\mathbf{S}}_{\alpha}}$$

Time-ordering  $\mathcal{T}$  due to noncommutativity of the spin operators

- solution: Wei-Norman-Kolokolov transformation

$$\mathcal{T} e^{i \int_0^t dt' \theta \hat{\mathbf{S}}} = e^{\pm \hat{\mathbf{S}}_{\mp} \psi_{\pm}(t)} e^{i \hat{\mathbf{S}}_z \int_0^t dt' \rho(t')} \exp \left[ i \hat{\mathbf{S}}_{\pm} \int_0^t dt' \psi_{\mp}(t') e^{\mp i \int_0^{t'} d\tau \rho(\tau)} dt' \right] e^{\mp \hat{\mathbf{S}}_{\mp} \psi_{\pm}(0)}$$

$$\theta_z = \rho - 2\psi_+ \psi_-, \quad \frac{\theta_x \mp i \theta_y}{2} = \psi_{\mp}, \quad \frac{\theta_x \pm i \theta_y}{2} = \mp i \psi_{\pm} + \rho \psi_{\pm} - \psi_{\mp} \psi_{\pm}^2, \quad \theta^2 = \rho^2 \mp 4i \psi_{\mp} \psi_{\pm}$$

$$\nu_{\sigma}(\varepsilon) = \frac{1 + e^{-\beta\varepsilon}}{2Z} \sum_{\alpha, n_{\uparrow}, n_{\downarrow}} \frac{\text{sh} \frac{\beta b(2m+1)}{2}}{\text{sh} \frac{\beta b}{2}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta \mathcal{E}_{n_{\uparrow}, n_{\downarrow}}} \left\{ \left[ \frac{Z_{n_{\downarrow}}(\epsilon_{\alpha})}{Z_{n_{\downarrow}}} - \frac{Z_{n_{\uparrow}}(\epsilon_{\alpha})}{(2m+1)Z_{n_{\uparrow}}} \right] \left[ 1 + B_m(\sigma m \beta b) \right] \right. \\ \times \delta(\varepsilon - \epsilon_{\alpha\sigma} + \mathcal{E}_{n_{\uparrow}, n_{\downarrow}} - \mathcal{E}_{n_{\uparrow}, n_{\downarrow}+1}) + \left[ \frac{Z_{n_{\uparrow}}(\epsilon_{\alpha})}{Z_{n_{\uparrow}}} + \frac{Z_{n_{\downarrow}}(\epsilon_{\alpha})}{(2m+1)Z_{n_{\downarrow}}} \right] \left[ 1 - B_{-m-1}(\sigma(m+1)\beta b) \right] \\ \left. \times \delta(\varepsilon - \epsilon_{\alpha\sigma} + \mathcal{E}_{n_{\uparrow}, n_{\downarrow}} - \mathcal{E}_{n_{\uparrow}+1, n_{\downarrow}}) \right\}, \quad Z = \sum_{n_{\uparrow}, n_{\downarrow} \in \mathbb{Z}} \frac{\text{sh} \frac{\beta b(2m+1)}{2}}{\text{sh} \frac{\beta b}{2}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta \mathcal{E}_{n_{\uparrow}, n_{\downarrow}}}.$$

where  $n = n_{\uparrow} + n_{\downarrow}$ ,  $m = (n_{\uparrow} - n_{\downarrow})/2$ ,

$$\mathcal{E}_{n_{\uparrow}, n_{\downarrow}} = E_c(n - N_0)^2 - Jm(m+1), \quad B_m(x) = \frac{2m+1}{2m} \text{cth} \left( \frac{2m+1}{2m} x \right) - \frac{1}{2m} \text{cth} \frac{x}{2m},$$

$$Z_n = \oint_{|z|=1} \frac{dz}{2\pi i} z^{-n-1} \prod_{\gamma} (1 + z e^{-\beta \epsilon_{\gamma}}), \quad Z_n(\epsilon_{\alpha}) = \oint_{|z|=1} \frac{dz}{2\pi i} z^{-n-1} \prod_{\gamma \neq \alpha} (1 + z e^{-\beta \epsilon_{\gamma}}).$$

- at  $b = J = 0$  the result coincides with Sedlmayr, Yurkevich, Lerner (2006)
- at  $b = 0$  the result for  $Z$  coincides with Alhassid, Rupp (2003)

- Fluctuations of thermodynamic DOS ( $\Delta^{-1} = dn/d\mu$ )

[Dyson (1962), Mehta, Dyson (1963), Efetov (1982)]

$$\langle \nu_0(E) [\nu_0(E + \omega) - \nu_0(E)] \rangle = \frac{1}{\delta^2} \left[ \delta(\omega/\delta) - R_{U/O}(\pi\omega/\delta) \right],$$

$$\nu_0(E) = \sum_{\alpha} \delta(E - \epsilon_{\alpha}), \quad \langle \nu_0(E) \rangle = \delta^{-1},$$

unitary ensemble:  $R_U(x) = \frac{\sin^2 x}{x^2}$

ortogonal ensemble:  $R_O(x) = \frac{\sin^2 x}{x^2} + \left( \frac{d}{dx} \frac{\sin x}{x} \right) \int_x^{\infty} \frac{\sin t}{t} dt$

at  $T \gg \delta$  fluctuations of  $\Delta$  are gaussian and

$$\frac{\overline{(\Delta - \delta)^2}}{\delta^2} = \frac{c\delta^2}{\beta T^2} \ll 1, \quad c \approx 0.02, \quad \beta_U = 2\beta_O = 2$$

common wisdom: Fluctuations of single-particle levels are not important at  $T \gg \delta$

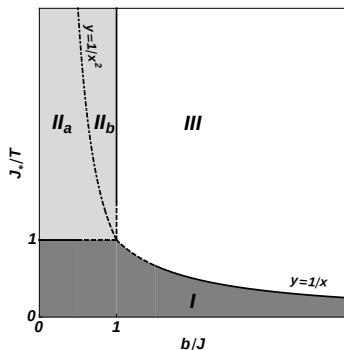
renormalized exchange energy  $\mathcal{J}$ :

$$\frac{1}{\mathcal{J}} = \frac{1}{J} - \frac{1}{\Delta}$$

Fluctuations of  $\Delta$  can lead to  $\mathcal{J} < 0$  (Stoner instability)

$$\frac{\overline{(\Delta - \delta)^2}}{\delta^4} \ll \frac{1}{J_\star^2} \equiv \left( \frac{1}{J} - \frac{1}{\delta} \right)^2, \quad \Rightarrow \quad J_\star \ll T$$

Near Stoner instability level fluctuations are not important at  $T \gg J_\star \gg \delta, J$



region

I:  $\frac{J_*}{T} \max\{1, \frac{b}{J}\} \ll 1$

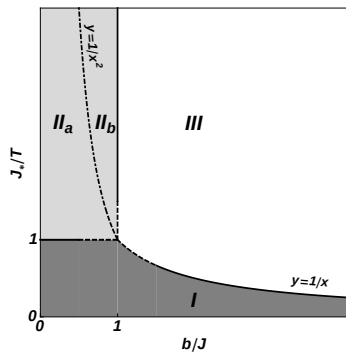
II<sub>a</sub>:  $\frac{b^2}{J^2} \ll \frac{T}{J_*} \ll 1$

II<sub>b</sub>:  $\frac{T}{J_*} \ll \frac{b^2}{J^2} \ll 1$

III:  $\frac{T}{J_*} \ll \frac{b}{J}, \frac{J}{b} \ll 1$

Averaged spin susceptibility  $\overline{\chi(T, b)}$

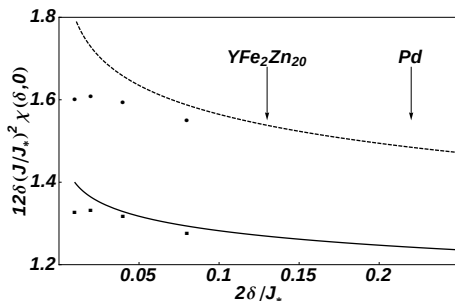
$$\begin{aligned} & \frac{J_*}{2J\delta} \left\{ 1 + \frac{J_*}{6T} + \frac{c}{\beta} \frac{J_*^2}{T^2} \left[ 1 + \frac{J_*}{2T} \right] - \frac{J_*^3 b^2}{120 T^3 J^2} \left[ 1 + \frac{10c}{\beta} \frac{J_*^2}{T^2} \right] \right\} \\ & \frac{T}{b^2} \left[ 1 - \frac{J_*^2 b^2}{4 J^2 T^2 \text{sh}^2(J_* b / 2 J T)} \right] + \frac{T}{2 \beta \pi^2} \frac{\partial^2}{\partial b^2} \left[ \frac{J_* b}{J T} \text{cth} \frac{J_* b}{2 J T} - \frac{J_*^2 b^2 [\ln(2 J_* / T) + \gamma]}{4 J^2 T^2 \text{sh}^2(J_* b / 2 J T)} \right] \\ & \frac{J_*}{2 J \delta} \left\{ 1 - \frac{2 J_*}{T} e^{-J_* b / J T} + \frac{2 J_*}{\beta \pi^2 T} \left( \ln \frac{b}{J} - \frac{3}{2} \right) e^{-J_* b / J T} \right\} \\ & \frac{J_*}{2 J \delta} \left( 1 - \frac{1}{\beta \pi^2} \frac{J^2}{b^2} \right) \end{aligned}$$



| region                   | $\overline{\chi(T, b)}$                                        |
|--------------------------|----------------------------------------------------------------|
| I, II <sub>b</sub> , III | Pauli-type: $\frac{J_\star}{2J\delta}$                         |
| II <sub>a</sub>          | Curie-type: $\frac{1}{12T} \left( \frac{J_\star}{J} \right)^2$ |

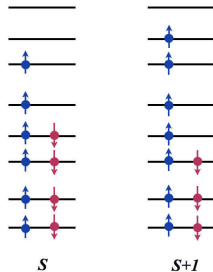
averaged zero-field spin susceptibility ( $T \gg J_* e^{-\beta\pi^2}$ ):

$$\overline{\chi(T, 0)} = \frac{1}{12T} \left( \frac{J_*}{J} \right)^2 \left[ 1 + \overbrace{\frac{1}{\beta\pi^2} \left( \ln \frac{2J_*}{T} + \gamma + 2 \right)}^{\text{effect of randomness}} \right],$$

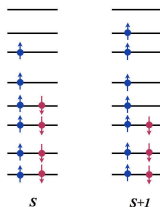


points are numerical simulations from Kurland, Aleiner, Altshuler (2000)

At  $T \ll J_*$  level fluctuations are important but numerically not large

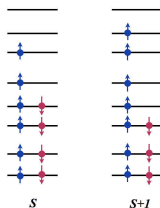


$$E_{S+1} - E_S = (2S + 1)\delta + \Delta E - J(2S + 2) = 0, \quad \Delta E = \Delta n_{2S}\delta$$



$$E_{S+1} - E_S = (2S + 1)\delta + \Delta E - J(2S + 2) = 0, \quad \Delta E = \Delta n_{2S}\delta$$

$$S \equiv \overline{S} + \Delta S = \frac{J_\star}{2J} \left[ 1 - \Delta n_{2S} \right] \implies \overline{\chi} = \frac{\overline{S^2}}{3T} = \left( \frac{J_\star}{2\delta} \right)^2 \left[ 1 + \overline{(\Delta n_{2S})^2} \right]$$



$$E_{S+1} - E_S = (2S + 1)\delta + \Delta E - J(2S + 2) = 0, \quad \Delta E = \Delta n_{2S}\delta$$

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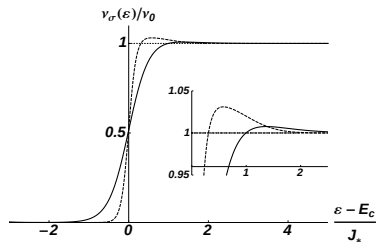
$$\overline{(\Delta n_{2S})^2} = \frac{2}{\beta\pi^2} \left( \ln \frac{J_\star}{\delta} + \text{const} \right) \implies \overline{\chi} = \frac{1}{3T} \left( \frac{J_\star}{2\delta} \right)^2 \left[ 1 + \frac{2}{\beta\pi^2} \left( \ln \frac{J_\star}{\delta} + \text{const} \right) \right]$$

c.f.

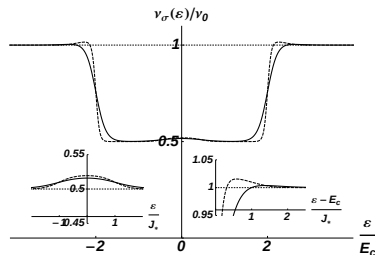
$$\overline{\chi(T, 0)} = \frac{1}{12T} \left( \frac{J_\star}{J} \right)^2 \left[ 1 + \frac{1}{\beta\pi^2} \left( \ln \frac{2J_\star}{T} + \gamma + 2 \right) \right]$$

$$\begin{aligned} \frac{\nu_\sigma(\varepsilon)}{\nu_0} &= \sum_{n,p=\pm} e^{-\beta E_c(n-N_0)^2} \left\{ f_F\left(p\varepsilon - 2p\Omega_n^{-p}\right) + \frac{\sqrt{\pi}\beta J e^{\frac{\beta}{4}(\sigma b - J_\star \frac{b^2+J^2}{J^2})}}{8\sqrt{J_\star} \operatorname{ch}(\frac{\beta b}{4}) \operatorname{sh}(\frac{J_\star \beta b}{2J})} \sum_{s=\pm} s \left[ \operatorname{erfi}\left(\frac{\sqrt{\beta J_\star}(sb - J)}{2J}\right) \right. \right. \\ &\quad \left. \left. \times f_F\left(p\varepsilon - 2p\Omega_n^{-p} + \frac{\sigma b}{2}\right) - \mathbb{F}\left(\beta\left[p\varepsilon - 2p\Omega_n^{-p} + \frac{\sigma b}{2}\right], \frac{sb}{2J}, \sqrt{\beta J_\star}\right) \right] \right\} \bigg/ \sum_n e^{-\beta E_c(n-N_0)^2}, \\ \Omega_n^p &= E_c(n - N_0 + p/2), \quad \operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt e^{t^2}, \quad \mathbb{F}(x, y, z) = \frac{1}{2} e^{-x/2} \int_{-\infty}^{\infty} \frac{dt e^{ixt}}{\operatorname{ch}(\pi t)} \operatorname{erfi}(z(y - it)) \end{aligned}$$

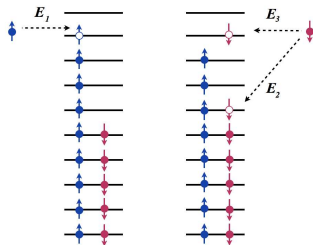
Tunneling DOS:  $\delta \ll T \ll J_*$  &  $b = 0$  (no level fluctuations)



Coulomb valley  
 $N_0$  integer

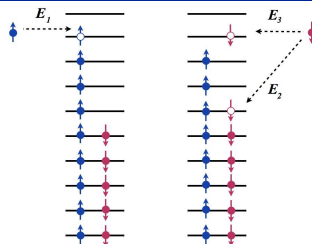


Coulomb peak  
 $N_0$  half-integer



transition  $n_{\uparrow} \rightarrow n_{\uparrow} + 1$

$$E_1 = E_c - J(S + 3/4) + \delta \simeq E_c - J_{\star}/2$$



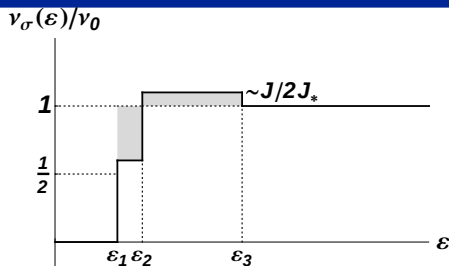
transition  $n_{\uparrow} \rightarrow n_{\uparrow} + 1$

$$E_1 = E_c - J(S + 3/4) + \delta \simeq E_c - J_{\star}/2$$

transition  $n_{\downarrow} \rightarrow n_{\downarrow} + 1$

$$E_2 = E_c + J(S + 1/4) - \delta(2S - 1) \simeq E_c - J_{\star}/2$$

$$E_3 = E_c + J(S + 1/4) + \delta \simeq E_c + J_{\star}/2$$



transition  $n_\uparrow \rightarrow n_\uparrow + 1$

$$E_1 = E_c - J(S + 3/4) + \delta \simeq E_c - J_*/2$$

transition  $n_\downarrow \rightarrow n_\downarrow + 1$

$$E_2 = E_c + J(S + 1/4) - \delta(2S - 1) \simeq E_c - J_*/2$$

$$E_3 = E_c + J(S + 1/4) + \delta \simeq E_c + J_*/2$$

width of the maximum  $\sim J_*$

strong spin-orbit coupling leads to Ising exchange:

$$\mathcal{H} = \sum_{\alpha, \sigma} \epsilon_{\alpha, \sigma} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} + E_c \left( \hat{N} - N_0 \right)^2 - J_z \hat{S}_z^2$$

[Alhassid, Rupp (2003); Usaj, Baranger (2005); Türeci, Alhassid (2006); Murthy (2008)]

the absence of mesoscopic Stoner instability

N.B. At intermediate spin-orbit coupling description in terms of universal hamiltonian is not possible

$$\mathcal{H} = \sum_{\alpha, \sigma} \epsilon_{\alpha, \sigma} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} + E_c \left( \hat{N} - N_0 \right)^2 - J_{\perp} [\hat{S}_x^2 + \hat{S}_y^2] - J_z \hat{S}_z^2$$

[ Waintal, Brouwer (2003); Usaj, Baranger (2005); Kiselev, Gefen (2006)]

results:

spin susceptibilities:

$$\chi_{zz} = \frac{\beta}{Z} \sum_{n_{\uparrow} n_{\downarrow}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta \mathcal{E}_{n_{\uparrow}, n_{\downarrow}}} \sum_{l=-m}^m l^2 e^{-\beta b l + \beta (J_z - J_{\perp}) l^2}$$

$$\chi_{\perp}(\omega) = -\frac{1}{Z} \sum_{n_{\uparrow} n_{\downarrow}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta \mathcal{E}_{n_{\uparrow}, n_{\downarrow}}} \sum_{l=-m}^m e^{\beta (J_z - J_{\perp}) l^2} \sum_{\sigma=\pm} \frac{\sigma e^{\sigma \beta b l} [m(m+1) - l^2 - l]}{\omega + b + \sigma(2l+1)(J_z - J_{\perp}) + i0}$$

where

$$Z = \sum_{n_{\uparrow} n_{\downarrow}} Z_{n_{\uparrow}} Z_{n_{\downarrow}} e^{-\beta \mathcal{E}_{n_{\uparrow}, n_{\downarrow}}} \sum_{l=-m}^m e^{-\beta b l + \beta (J_z - J_{\perp}) l^2}, \quad \mathcal{E}_{n_{\uparrow}, n_{\downarrow}} = E_c (n - N_0)^2 - J_{\perp} m(m+1)$$

- At  $J_{\perp} = 0$  results coincide with Nissan-Cohen, Gefen, Kiselev, Lerner (2011)

- exact analytical results for the spin susceptibility and TDOS in a QD with direct Coulomb and exchange interactions in the presence of Zeeman splitting
  - nonmonotonicity in the TDOS at  $J \lesssim \delta \ll T \ll J_*$
- level fluctuations
  - the zero-field spin susceptibility at  $T \ll J_*$
  - are suppressed by weak magnetic field  $b \gg J$
- extensions and future work: spin susceptibility and TDOS
  - for  $XXZ$  and  $XYZ$  anisotropic exchange
  - in the presence of Cooper-channel interaction

details can be found in

- I.S.B., Y. Gefen, M. Kiselev, JETP Letters 92, 766 (2010)
- I.S.B., Y. Gefen, M. Kiselev, Phys. Rev. B 85, 155331 (2012)

$$\mathcal{H} = \sum_{\alpha,\sigma} \epsilon_{\alpha,\sigma} a_{\alpha,\sigma}^\dagger a_{\alpha,\sigma} + E_c \left( \hat{N} - N_0 \right)^2 - J \hat{\mathbf{S}}^2 \\ + \sum_{\alpha,\beta,\sigma} a_{\alpha,\sigma}^\dagger a_{\beta,\sigma} \left( \hat{N} - N_0 \right) X_{\alpha\beta} + \sum_{\alpha,\beta,\gamma,\delta,\sigma,\sigma'} Y_{\alpha\beta\gamma\delta} a_{\alpha,\sigma}^\dagger a_{\beta,\sigma'}^\dagger a_{\gamma,\sigma'} a_{\delta,\sigma}$$

where

$$\langle X_{\alpha\beta} X_{\alpha\beta} \rangle \sim \frac{\delta^3}{E_{\text{Th}}}, \quad \langle Y_{\alpha\beta\gamma\delta} \rangle \sim \frac{\delta^2}{E_{\text{Th}}} \left( \delta_{\alpha\delta} \delta_{\beta\gamma} + \delta_{\alpha\beta} \delta_{\gamma\delta} \right), \quad \langle [Y_{\alpha\beta\gamma\delta} - \langle Y_{\alpha\beta\gamma\delta} \rangle]^2 \rangle \sim \frac{\delta^4}{E_{\text{Th}}^2}$$

[Blanter, Mirlin, Muzykantskii (1997); Altshuler, Gefen, Kamenev, Levitov (1997)]

[for review, see Aleiner, Brouwer, Glazman (2002); Ullmo (2008)]

$$\mathcal{H} = \sum_{\alpha, \sigma} \varepsilon_{\alpha} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} + E_c \left( \hat{N} - N_0 \right)^2 - J \hat{\mathbf{S}}^2,$$

$$\hat{N} = \sum_{\alpha, \sigma} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma}, \quad \hat{\mathbf{S}}_{\sigma \sigma'} = \frac{1}{2} \sum_{\alpha, \sigma} a_{\alpha, \sigma}^{\dagger} \vec{\sigma}_{\sigma \sigma'} a_{\alpha, \sigma'}$$

- ▶ Functional integral in imaginary time
- ▶ Decoupling Coulomb interaction  $E_c (\hat{N} - N_0)^2$  by the Hubbard-Stratonovich field  $\phi$

$$Z = \int_{-\pi T}^{\pi T} d\phi_0 \mathcal{D}(\tau_1, \tau_1 | \phi_0) \tilde{Z}(\phi_0)$$

Charge problem:

$$\mathcal{D}(\tau_1, \tau_2 | \phi_0) = \sum_{m \in \mathbb{Z}} \int \mathcal{D}[\tilde{\phi}] e^{-\int_0^{\beta} d\tau \frac{\tilde{\phi}^2(\tau)}{4E_c} - i \int_{\tau_1}^{\tau_2} d\tau \tilde{\phi}(\tau) - \frac{\pi^2 T}{E_c} \left( m + \frac{\beta \phi_0}{2\pi} \right)^2 + 2\pi i N_0 \left( m + \frac{\beta \phi_0}{2\pi} \right) - i 2\pi m T (\tau_1 - \tau_2)}$$

Spin problem:

$$\tilde{Z}(\phi_0) = \text{Tr} e^{-\beta \mathcal{H}_J}, \quad \mathcal{H}_J = \sum_{\alpha, \sigma} \tilde{\varepsilon}_{\alpha} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} - J \hat{\mathbf{S}}^2 \quad \tilde{\varepsilon}_{\alpha} = \varepsilon_{\alpha} - i\phi_0$$

N.B.: Charge and spin are entangled due to integration over  $\phi_0$ !

- The Hubbard-Stratonovich transformation of exchange interaction:

$$e^{\mp i t J \hat{\mathbf{S}}^2} = \lim_{N \rightarrow \infty} \prod_{\alpha} \prod_{n=1}^N \int d\theta_n e^{\pm \frac{i}{4J} t \theta_n^2 / N} e^{i t \theta_n \hat{\mathbf{S}}_{\alpha} / N} = \prod_{\alpha} \int \mathcal{D}[\theta] e^{\pm \frac{i}{4J} \int_0^t dt' \theta^2} \mathcal{T} e^{i \int_0^t dt' \theta \hat{\mathbf{S}}_{\alpha}}$$

Time-ordering  $\mathcal{T}$  due to noncommutativity of the spin operators!

- Wei-Norman-Kolokolov transformation

$$\mathcal{T} e^{i \int_0^t dt' \theta \hat{\mathbf{S}}} = e^{\pm \hat{\mathbf{S}}_{\mp} \psi_{\pm}(t)} e^{i \hat{\mathbf{S}}_z \int_0^t dt' \rho(t')} \exp \left[ i \hat{\mathbf{S}}_{\pm} \int_0^t dt' \psi_{\mp}(t') e^{\mp i \int_0^{t'} d\tau \rho(\tau)} dt' \right] e^{\mp \hat{\mathbf{S}}_{\mp} \psi_{\pm}(0)}$$

$$\theta_z = \rho - 2\psi_+ \psi_-, \quad \frac{\theta_x \mp i\theta_y}{2} = \psi_{\mp}, \quad \frac{\theta_x \pm i\theta_y}{2} = \mp i\psi_{\pm} + \rho\psi_{\pm} - \psi_{\mp}\psi_{\pm}^2, \quad \theta^2 = \rho^2 \mp 4i\psi_{\mp}\psi_{\pm}$$

N.B.: The Jacobian of transformation from  $\theta$  to  $\rho, \psi_{\pm}$  is  $\mathcal{J} = \exp \left[ \frac{i}{2} \int_0^t dt' \rho(t') \right]$

Initially,  $\theta_{x,y,z}$  are real variables, but now  $(\theta_x - i\theta_y)^* \neq \theta_x + i\theta_y$

We impose constraints  $\psi_+ = \psi_-^*$  and  $\rho = -\rho^*$

- ▶ Two sets of variables:  $\theta_{1,2} \implies$  two sets of new variables  $\rho_{1,2}$  and  $\psi_{1,2}^\pm$
- ▶ We choose initial condition  $\psi_1^+(0) = \psi_2^-(0) = 0$
- ▶ Exact integration over  $\psi_{1,2}^\pm$
- ▶ New variables:  $\rho_{1,2}(t) = \mp i \dot{\xi}_{1,2}$ ,  $\xi_1(0) = \xi_2(0)$ ,  $\xi_1(t_1) + \xi_2(t_2) = 0$ ,

$$\begin{aligned} \tilde{Z}[\phi_0] = & \prod_{\gamma} \left( - \oint_{|z|=1} \frac{dz_{\gamma}}{2\pi i z_{\gamma}^2} e^{-z_{\gamma} [1 + e^{-2\beta \bar{\varepsilon} \gamma}]} \right) \int_0^{\infty} \frac{dy}{4y^d} e^{-y} \int_{-\infty}^{\infty} d\xi_1 d\xi_2 \prod_{\gamma} e^{-2z_{\gamma} e^{-\beta \bar{\varepsilon} \gamma} \cosh \frac{\xi_1 - \xi_2}{2}} \\ & \times \delta \left( \xi_1 + \xi_2 + 2 \ln \left[ 4y \sum_{\gamma} z_{\gamma} e^{-\beta \bar{\varepsilon} \gamma} \right] \right) \langle \xi_1 | e^{-i\mathcal{H}_0 t_1} e^{-\xi} e^{i\mathcal{H}_0 (t_1 + i\beta)} | \xi_2 \rangle \end{aligned}$$

$$\mathcal{H}_0 = -J \frac{\partial^2}{\partial \xi^2} + \frac{J}{4} e^{-\xi}, \quad E_{\nu} = J\nu^2, \quad \Psi_{\nu}(\xi) = \frac{2}{\pi} \sqrt{\nu \sinh 2\pi\nu} K_{2i\nu}(e^{-\xi/2})$$

$$\tilde{Z}[\phi_0] = \int_{-\infty}^{\infty} dh \, h \sinh(\beta h) e^{-\beta h^2/J} \prod_{\gamma} \left(1 + e^{-\beta(\tilde{\epsilon}_{\gamma} + h)}\right) \left(1 + e^{-\beta(\tilde{\epsilon}_{\gamma} - h)}\right)$$

► The grand partition function

$$\begin{aligned} Z &= \frac{1}{\sqrt{\pi\beta J}} e^{-\beta J/4} e^{-\beta b^2/4J} \sum_{n \in \mathbb{Z}} e^{-\beta E_c(n - N_0)^2} \int_{-\pi T}^{\pi T} \frac{d\phi_0}{2\pi T} e^{i\beta\phi_0 n} \\ &\times \int_{-\infty}^{\infty} dh \sinh(h) \frac{\sinh(bh/J)}{\sinh(\beta b/2)} e^{-Th^2/J} \prod_{\sigma=\pm} e^{-\beta\Omega_0(\mu - i\phi_0 + Th\sigma)} \\ e^{-\beta\Omega_0(\mu)} &= \prod_{\alpha} \left(1 + e^{-\beta(\epsilon_{\alpha} - \mu)}\right) \end{aligned}$$

$$\sum_{\sigma} \beta\Omega_0(\mu - i\phi_0 + Th\sigma) = 2\beta\Omega_0(\mu - i\phi_0) - \frac{Th^2}{\delta} + F(h),$$

$$F(h) = - \int_{-\infty}^{\infty} dE \, \Delta\nu_0(E) \ln \left[ 1 + \frac{\cosh h - 1}{2 \cosh^2(E/2T)} \right]$$