

Multifractality of wave functions: Interplay with interaction, classification, and symmetries

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Plan

- Introduction: Anderson localization and multifractality
- Multifractality and interaction
 - Dephasing and temperature scaling at localization transitions
Burmistrov, Bera, Evers, Gornyi, ADM, Annals Phys. 326, 1457 (2011)
 - Enhancement of superconductivity by Anderson localization
Burmistrov, Gornyi, ADM, PRL 108, 017002 (2012)
- Classification of composite operators
and symmetry properties of scaling dimensions
Gruzberg, Ludwig, ADM, Zirnbauer, PRL 107, 086403 (2011);
Gruzberg, ADM, Zirnbauer, to be published

Anderson localization



Philip W. Anderson

1958 “Absence of diffusion
in certain random lattices”

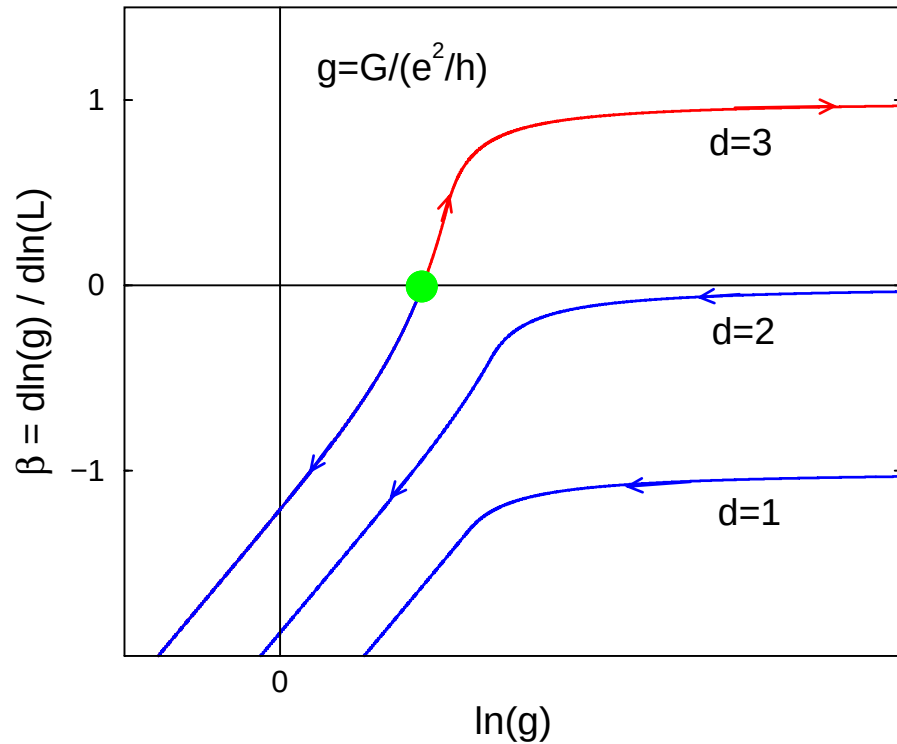
sufficiently strong disorder $\longrightarrow$ quantum localization

$\longrightarrow$ eigenstates exponentially localized, no diffusion

$\longrightarrow$ Anderson insulator

Nobel Prize 1977

Anderson transition

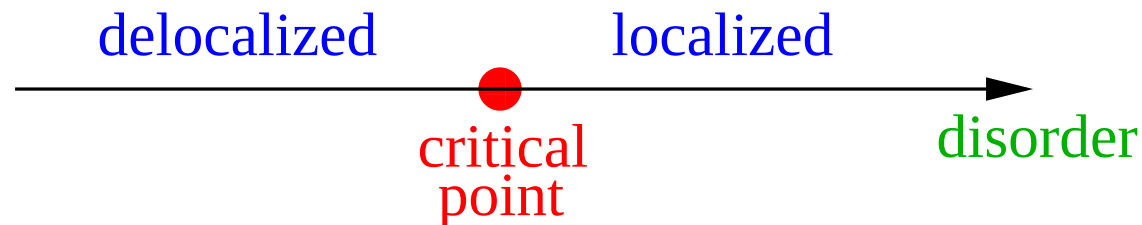


Scaling theory of localization:
Abrahams, Anderson, Licciardello,
Ramakrishnan '79

Modern approach:
RG for field theory (σ -model)

quasi-1D, 2D: metallic $\rightarrow$ localized crossover with increasing L

$d > 2$: metal-insulator transition



review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Field theory: non-linear σ -model

$$S[Q] = \frac{\pi\nu}{4} \int d^d \mathbf{r} \, \text{Tr} [-D(\nabla Q)^2 - 2i\omega \Lambda Q], \quad Q^2(\mathbf{r}) = 1$$

Wegner '79

σ -model manifold: symmetric space

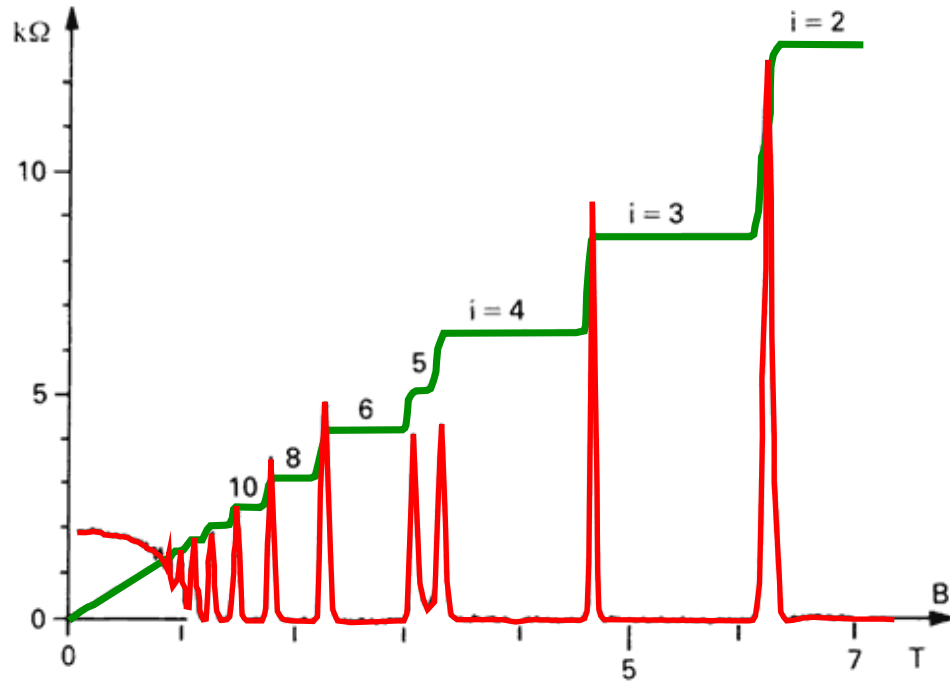
e.g. for broken time-reversal invariance:

$$\text{U}(2n)/\text{U}(n) \times \text{U}(n), \quad n \rightarrow 0$$

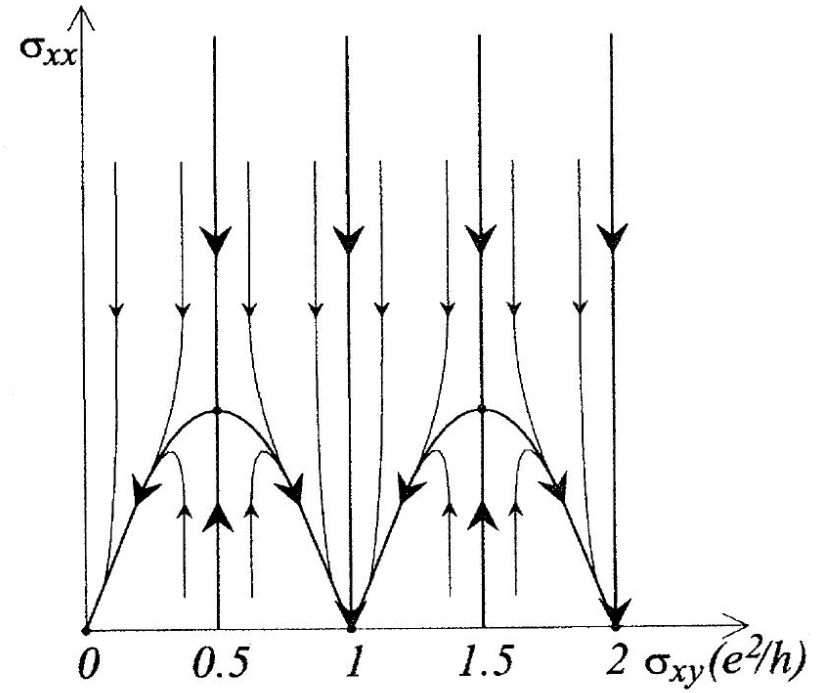
with Coulomb interaction: Finkelstein'83

supersymmetry (non-interacting systems): Efetov'82

Anderson localization & topology: Integer Quantum Hall Effect



von Klitzing '80 ; Nobel Prize '85



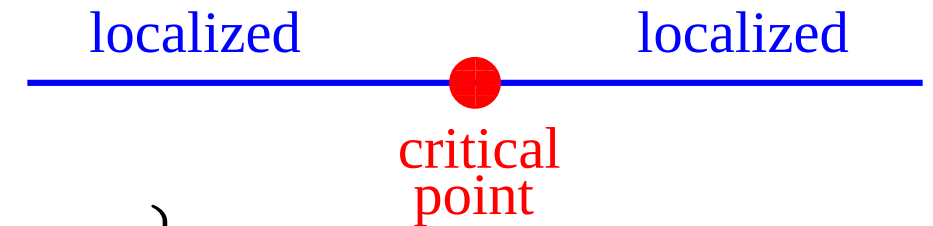
IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84

Field theory (Pruisken):

σ -model with topological term

$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \text{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \text{Tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$



QH insulators $\longrightarrow n = \dots, -2, -1, 0, 1, 2, \dots$ protected edge states

$\longrightarrow \mathbb{Z}$ topological insulator

Multifractality at the Anderson transition

$$P_q = \int d^d r |\psi(\mathbf{r})|^{2q} \quad \text{inverse participation ratio}$$

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

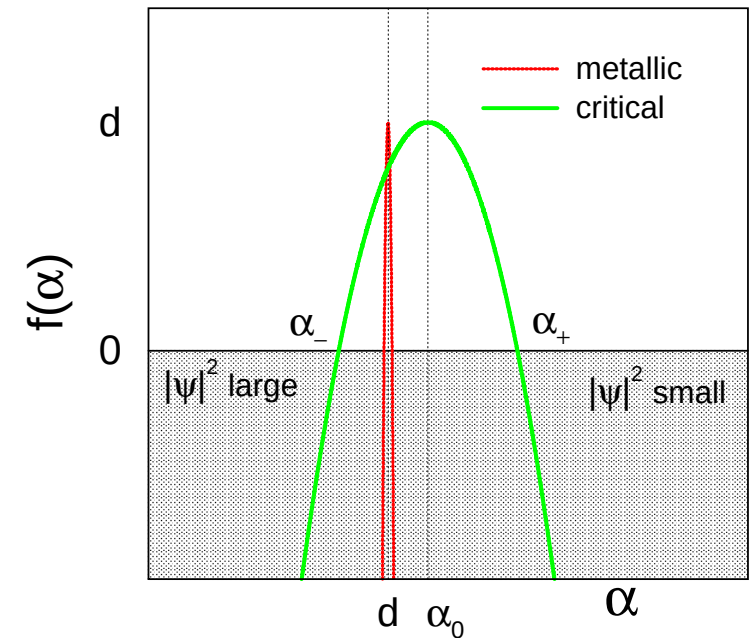
$$\tau_q = \underbrace{d(q-1)}_{\text{normal}} + \underbrace{\Delta_q}_{\text{anomalous}} \equiv D_q(q-1) \quad \text{multifractality}$$

$$\begin{aligned} \tau_q &\longrightarrow \text{Legendre transformation} \\ &\longrightarrow \text{singularity spectrum } f(\alpha) \end{aligned}$$

wave function statistics:

$$\mathcal{P}(\ln |\psi^2|) \sim L^{-d+f(\ln |\psi^2|/\ln L)}$$

$$L^{f(\alpha)} - \text{measure of the set of points where } |\psi|^2 \sim L^{-\alpha}$$



Multifractality (cont'd)

- Multifractality implies very **broad distribution** of observables characterizing wave functions. For example, parabolic $f(\alpha)$ implies log-normal distribution

$$\mathcal{P}(|\psi^2|) \propto \exp\{-\# \ln^2 |\psi^2| / \ln L\}$$

- field theory language: Δ_q – **scaling dimensions of operators**
 $\mathcal{O}^{(q)} \sim (Q\Lambda)^q$ **Wegner '80**

- Infinitely many operators with **negative scaling dimensions**, i.e. RG relevant (increasing under renormalization)

- 2-, 3-, 4-, ...-point **wave function correlations** at criticality

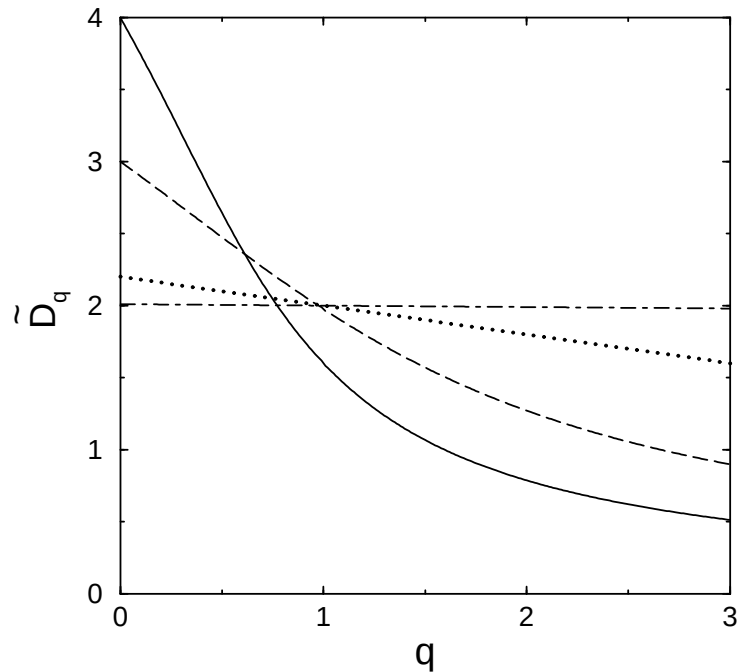
$$\langle |\psi_i^2(r_1)| |\psi_j^2(r_2)| \dots \rangle$$

also show power-law scaling controlled by multifractality

- **boundary multifractality**

Subramaniam, Gruzberg, Ludwig, Evers, Mildenberger, ADM, PRL'06

Dimensionality dependence of multifractality



Analytics ($2 + \epsilon$, one-loop) and
numerics

$$\tau_q = (q - 1)d - q(q - 1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d + \epsilon - \alpha)^2 / 4\epsilon + O(\epsilon^4)$$

$d = 4$ (full)

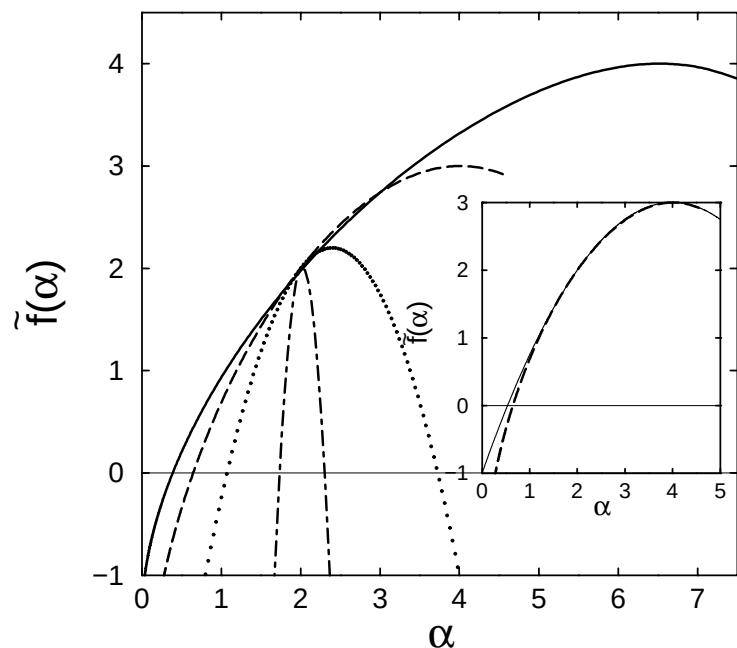
$d = 3$ (dashed)

$d = 2 + \epsilon$, $\epsilon = 0.2$ (dotted)

$d = 2 + \epsilon$, $\epsilon = 0.01$ (dot-dashed)

Inset: $d = 3$ (dashed)

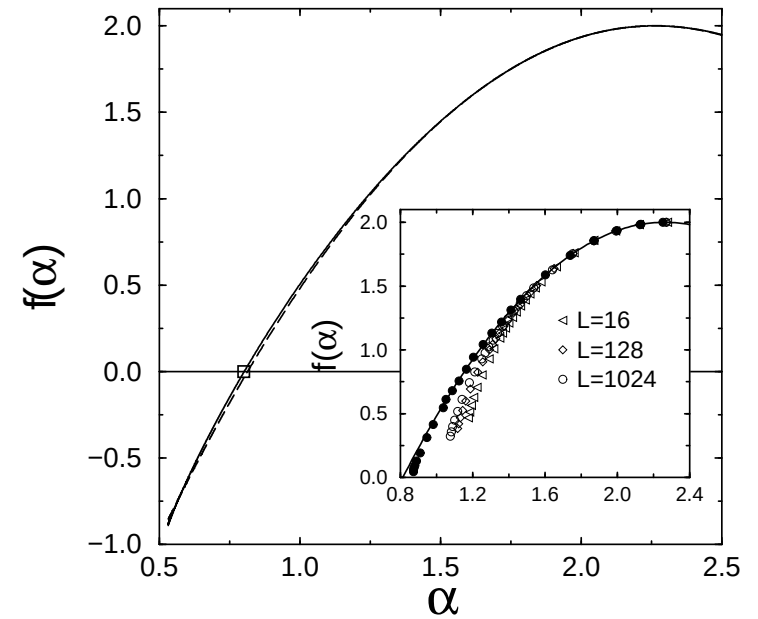
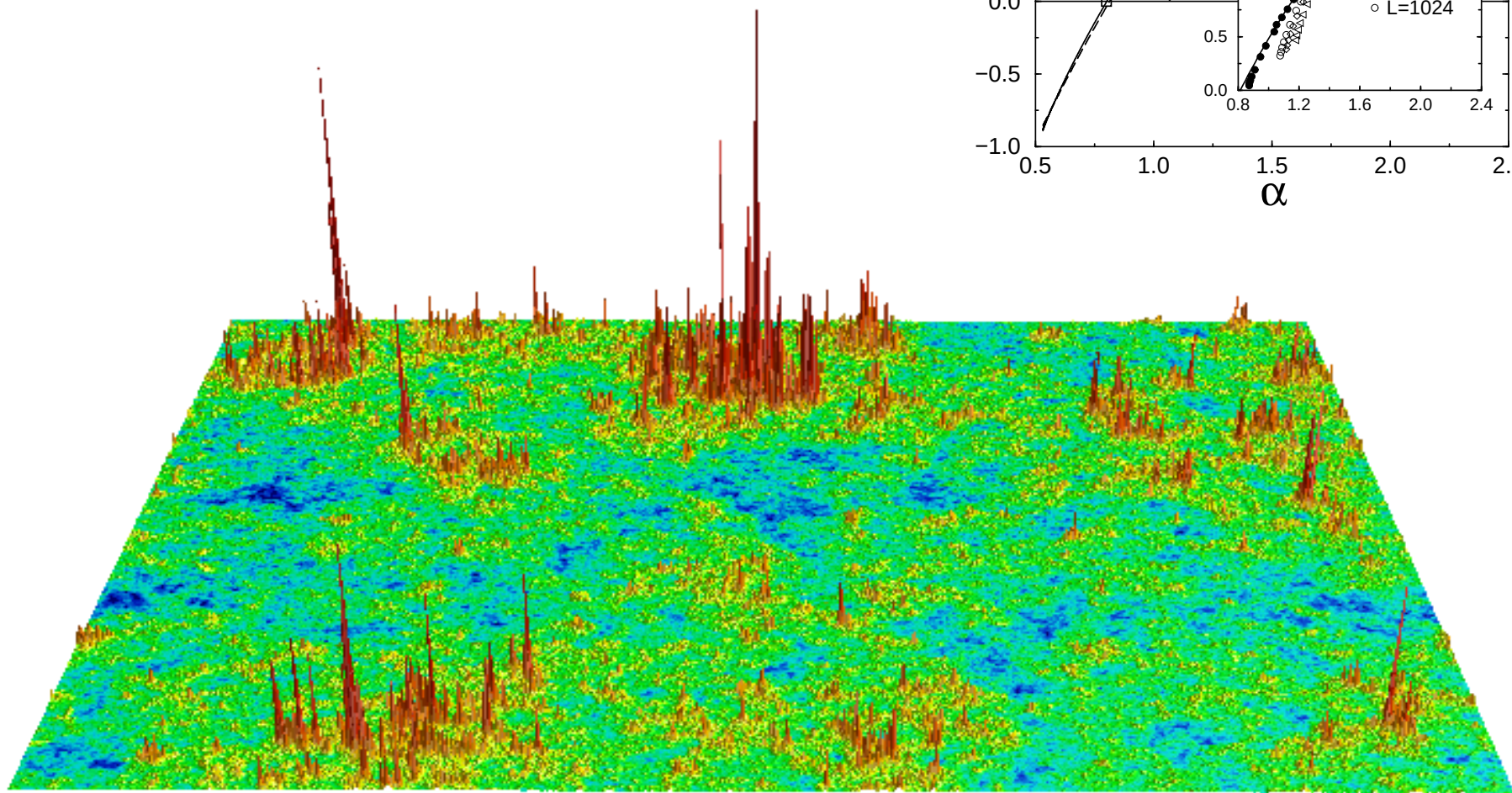
vs. $d = 2 + \epsilon$, $\epsilon = 1$ (full)



Mildenberger, Evers, ADM '02

Multifractality at the Quantum Hall transition

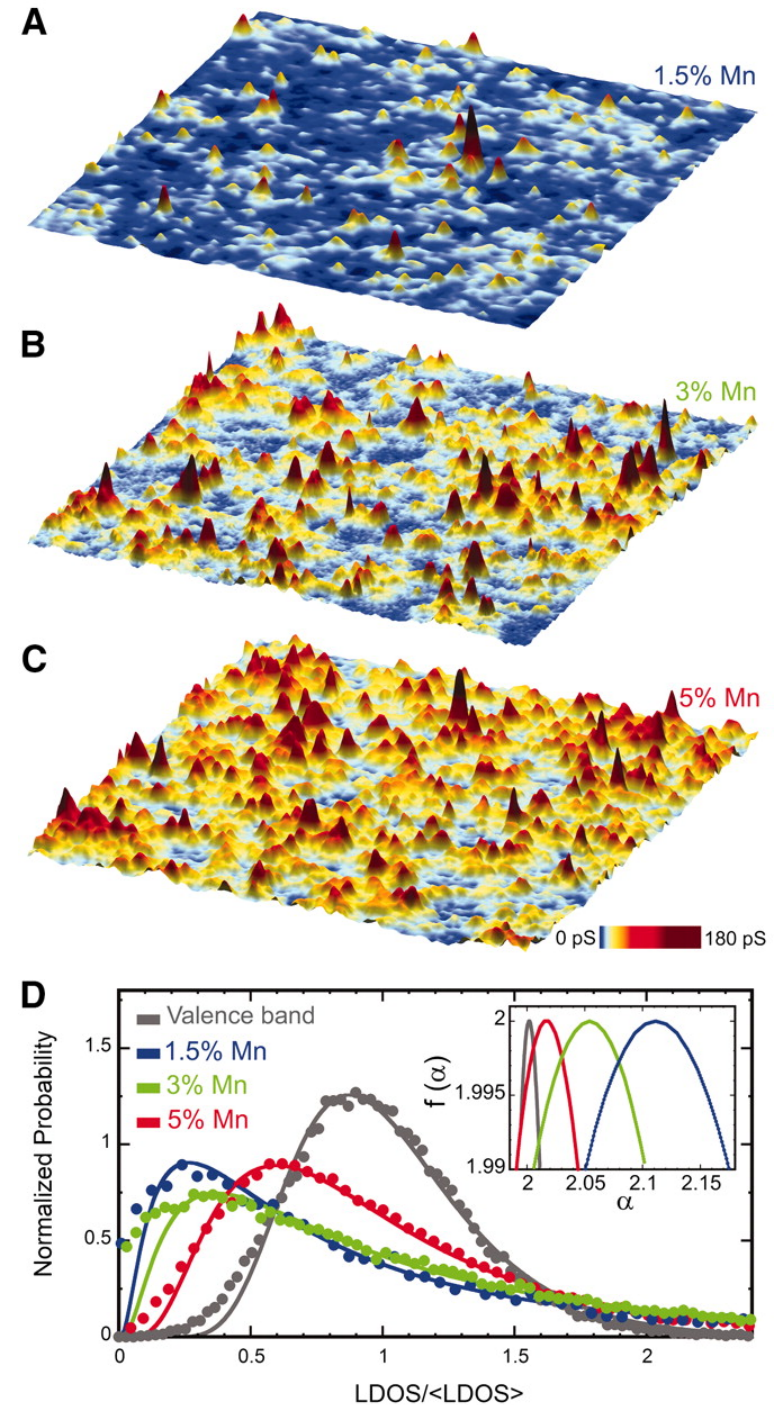
Evers, Mildenberger, ADM '01



Multifractality: Experiment I

Local DOS fluctuations
near metal-insulator transition
in $\text{Ga}_{1-x}\text{Mn}_x\text{As}$

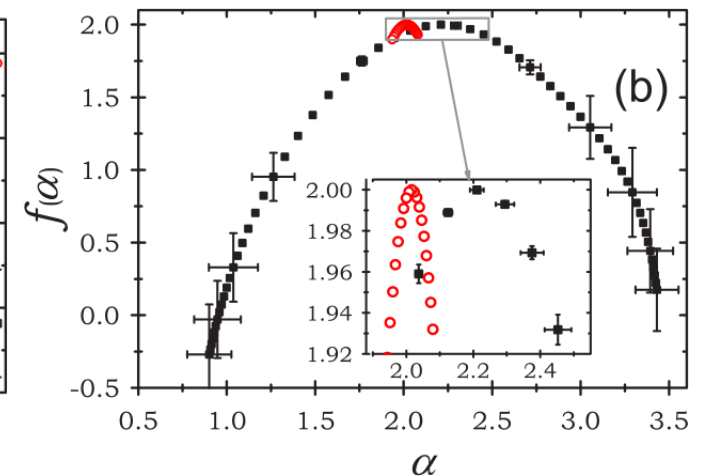
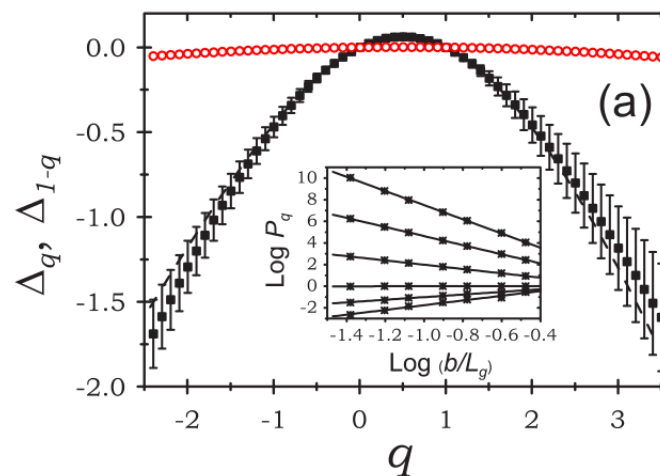
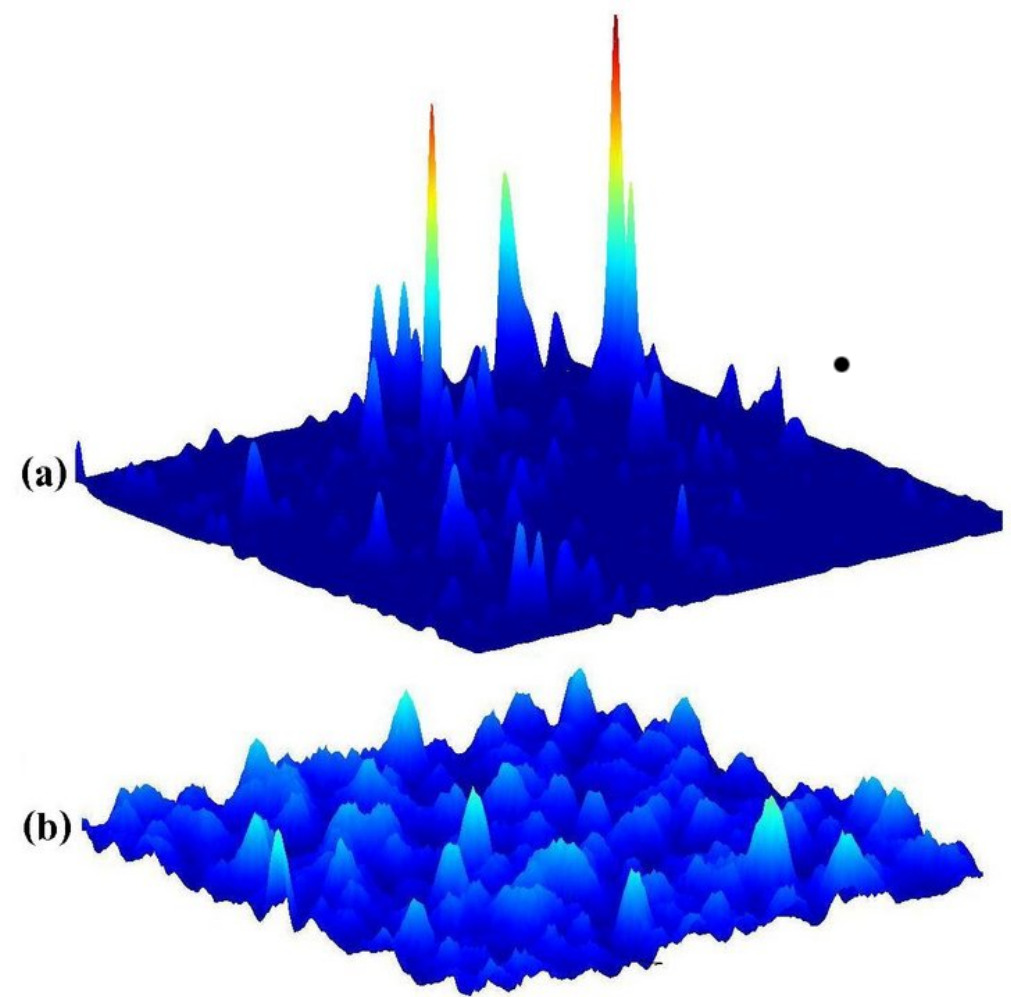
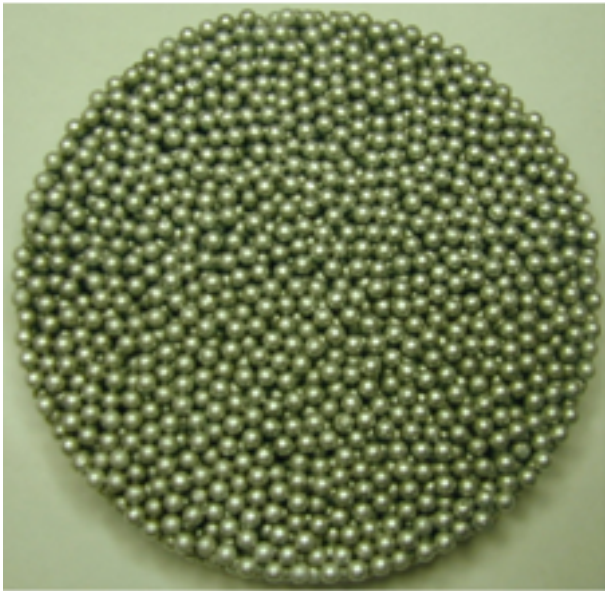
Richardella,...,Yazdani, Science '10



Multifractality: Experiment II

Ultrasound speckle in a system
of randomly packed Al beads

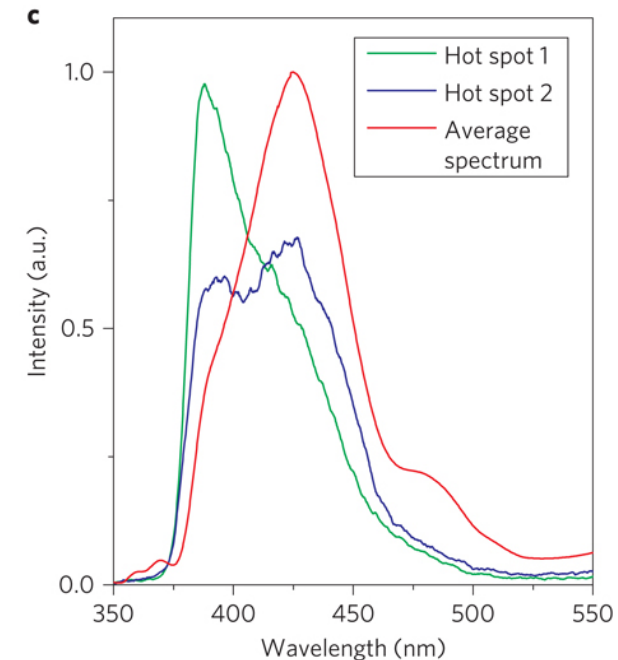
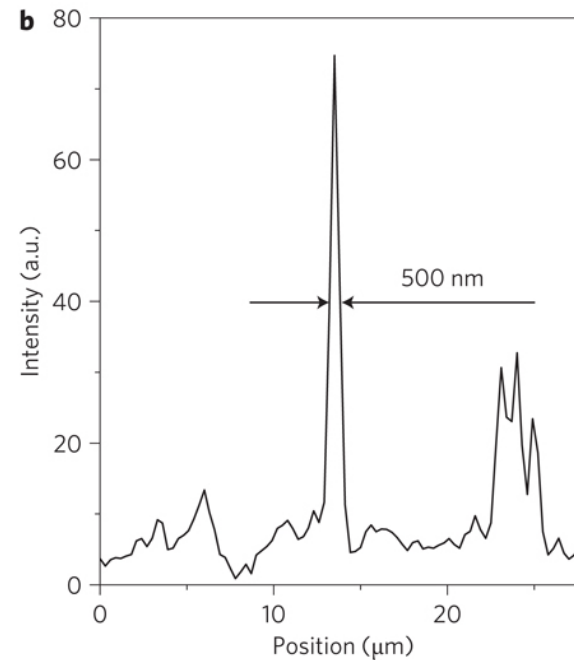
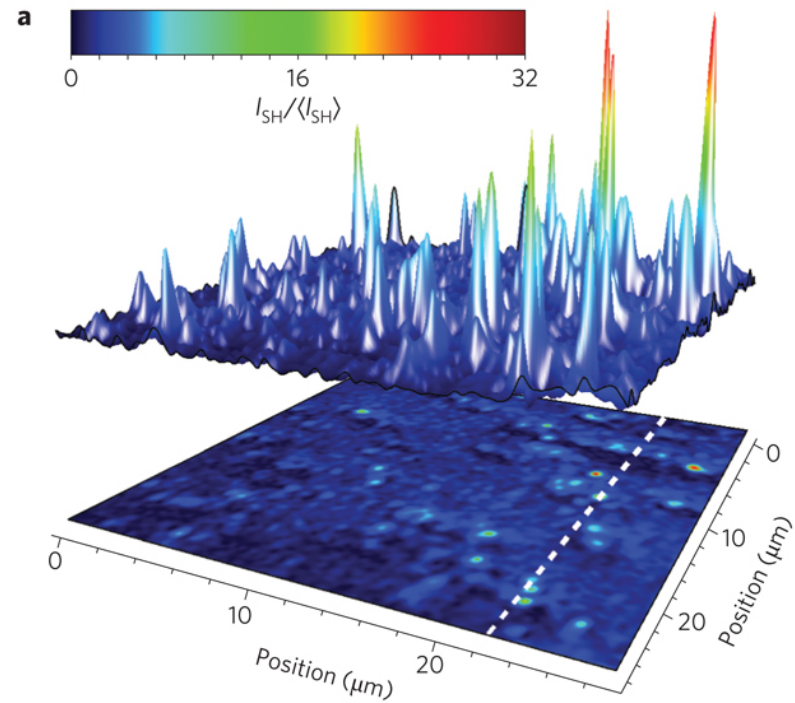
Faez, Strybulevich, Page,
Legendijk, van Tiggelen, PRL'09



Multifractality: Experiment III

Localization of light
in an array of dielectric
nano-needles

Mascheck et al,
Nature Photonics '12



Dephasing at metal-insulator and quantum Hall transitions

Burmistrov, Bera, Evers, Gornyi, ADM, Annals Phys. 326, 1457 (2011)

e-e interaction $\longrightarrow$ dephasing at finite T
 $\longrightarrow$ smearing of the transition

local. length $\xi \propto |n - n_c|^{-\nu}$, dephasing length $L_\phi \propto T^{-1/z_T}$
 $\longrightarrow$ transition width $\delta n \propto T^\kappa$, $\kappa = 1/\nu z_T$

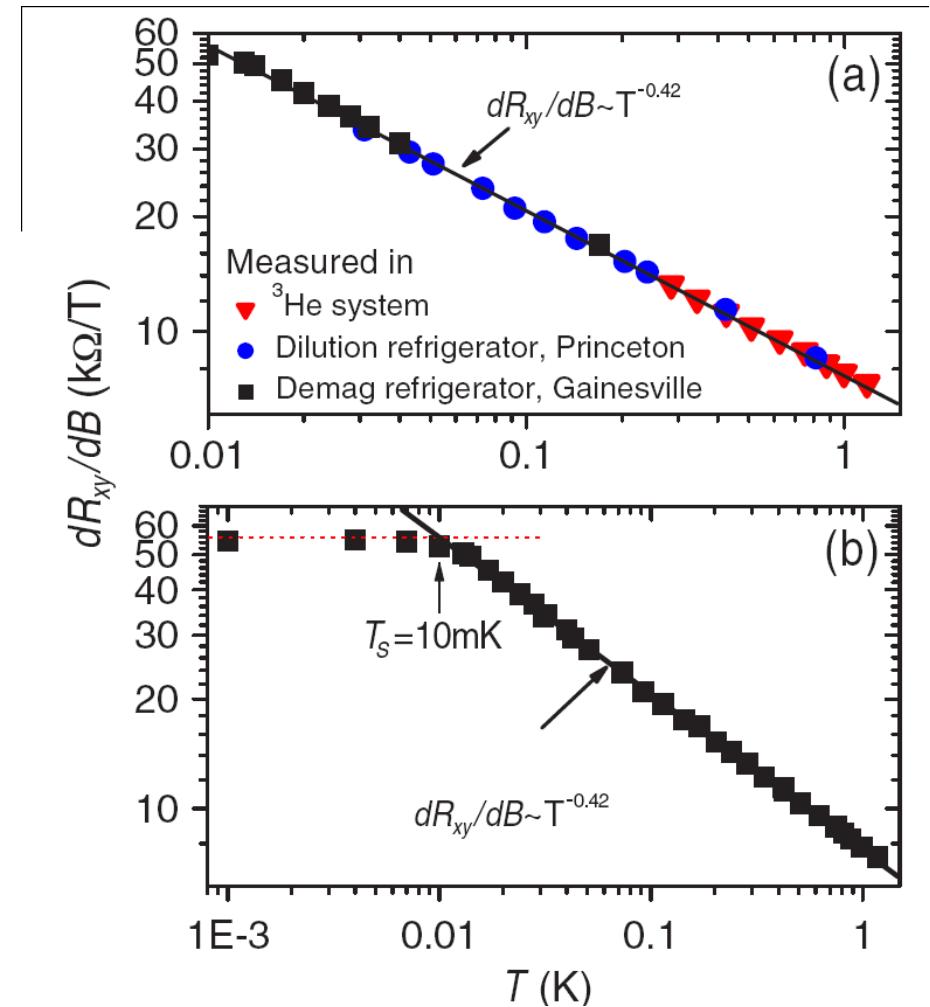
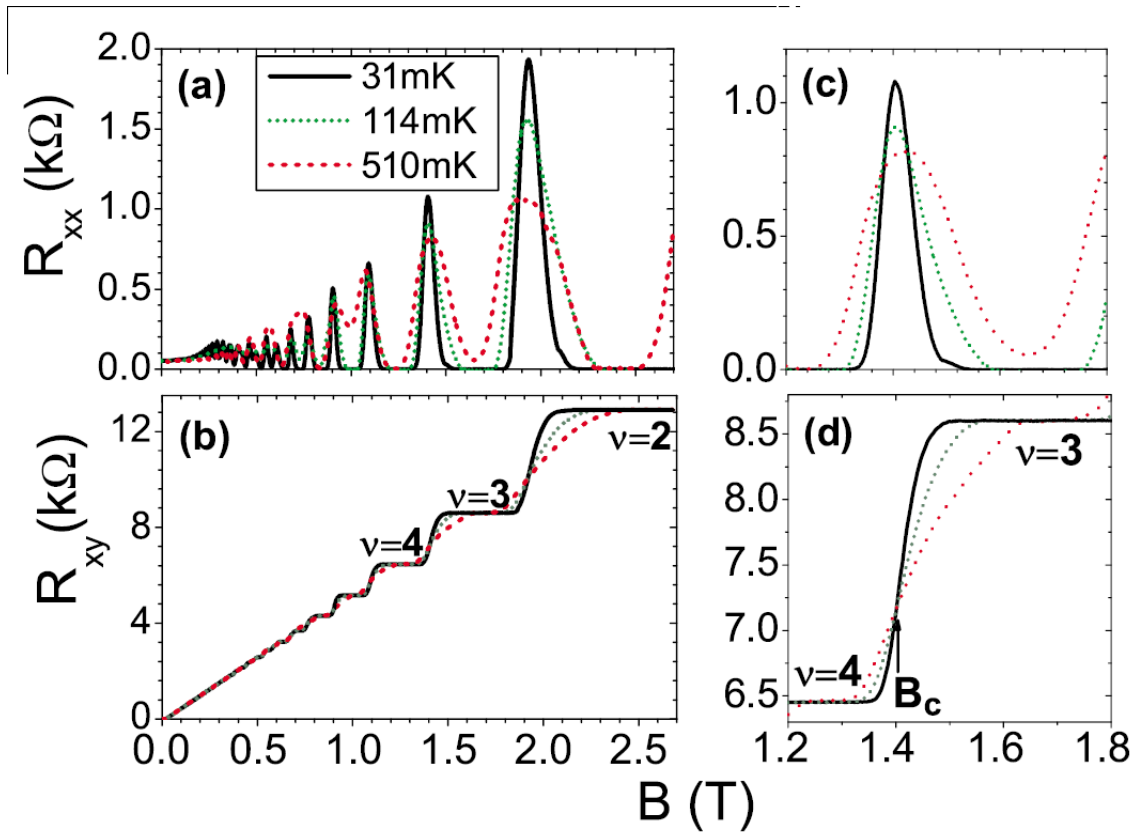
We focus on **short-range** e-e interaction:

- long-range Coulomb interaction negligible because of large dielectric constant
- 2D: screening by metallic gate
- interacting neutral particles (e.g. cold atoms)

Earlier works:

Lee, Wang, PRL'96; Wang, Fisher, Girvin, Chalker, PRB '00

Temperature scaling of quantum Hall transition



Transition width exponent

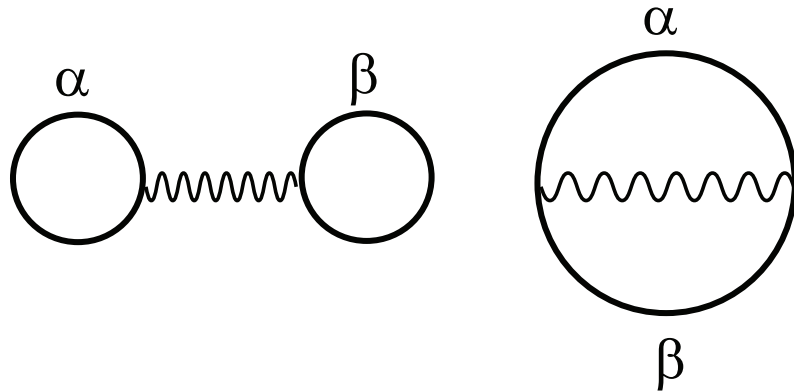
$$\kappa = 1/\nu z_T = 0.42 \pm 0.01$$

Wei, Tsui, Paalanen, Pruisken, PRL'88 ; Li et al., PRL'05, PRL'09

Interaction scaling at criticality

$$\mathcal{K}_1 = \frac{\Delta^2}{2} \sum_{\alpha\beta} \left\langle |\mathcal{B}_{\alpha\beta}(r_1, r_2)|^2 \delta(E + \omega - \epsilon_\alpha) \delta(E - \epsilon_\beta) \right\rangle$$

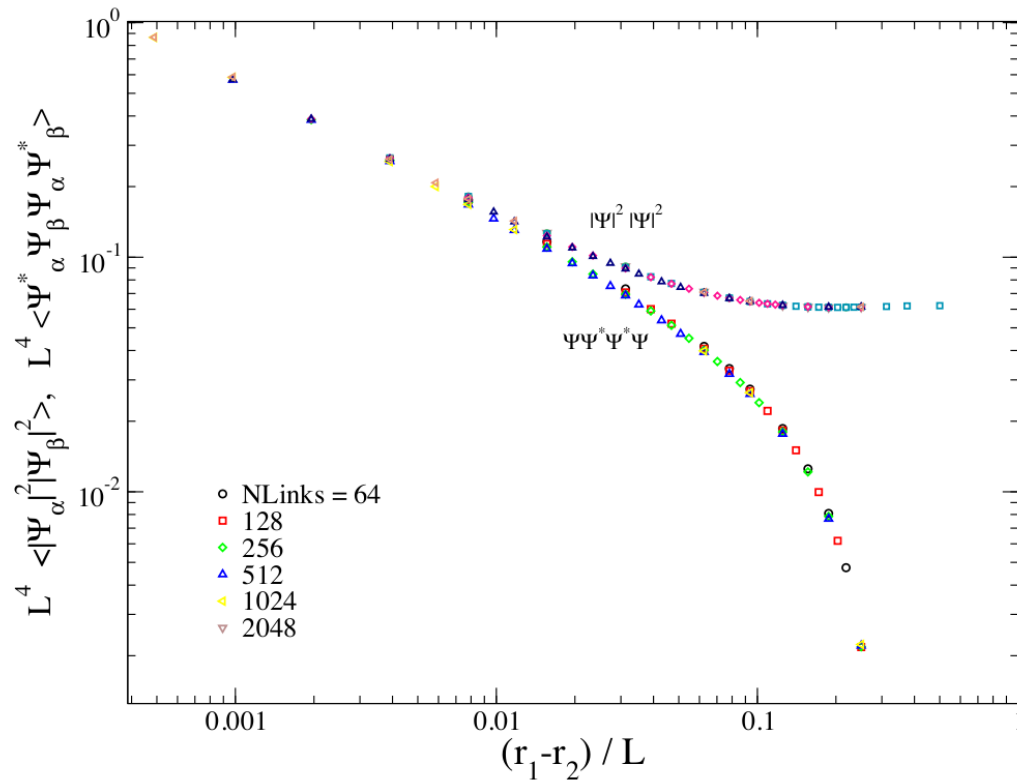
$$\mathcal{B}_{\alpha\beta}(r_1, r_2) = \phi_\alpha(r_1)\phi_\beta(r_2) - \phi_\alpha(r_2)\phi_\beta(r_1)$$



$$\mathcal{K}_1(r_1, r_2, E, \omega) = L^{-2d} \left(\frac{|r_1 - r_2|}{L_\omega} \right)^{\mu_2}, \quad |r_1 - r_2| \ll L_\omega$$

$$L_\omega = L(\Delta/|\omega|)^{1/d} \quad \text{length scale set by frequency } \omega$$

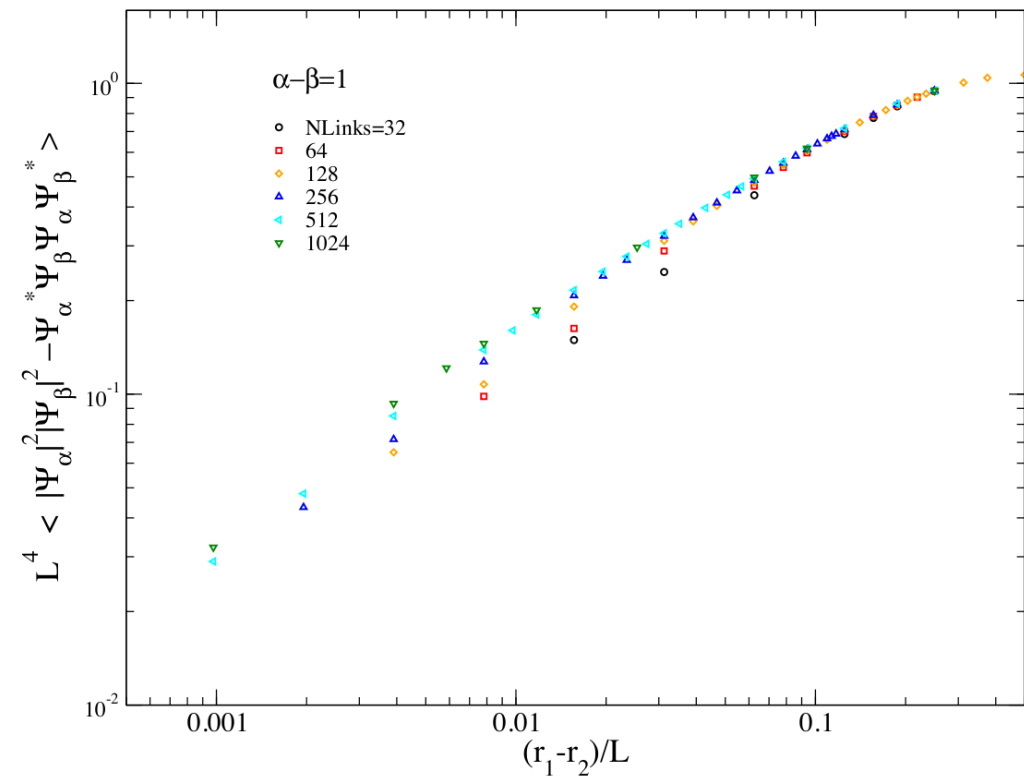
Interaction scaling at quantum Hall critical point



Hartree, Fock

enhanced by multifractality

exponent $\Delta_2 \simeq -0.52 < 0$

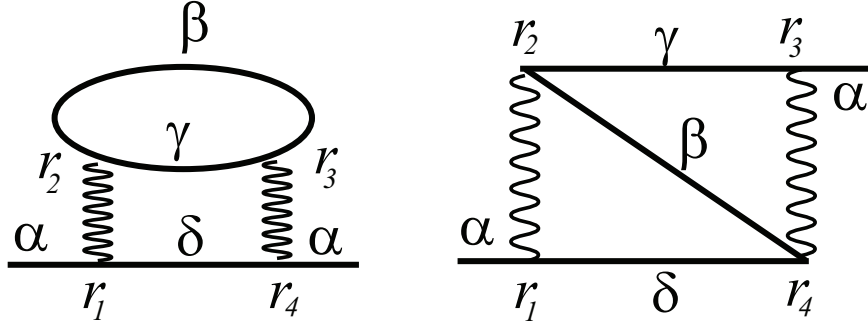


Hartree – Fock

suppressed by multifractality

exponent $\mu_2 \simeq 0.62 > 0$

Interaction-induced dephasing



$$\text{Im}\Sigma^R(0, 0) \sim -\frac{1}{2\Delta^3} \left(\prod_{j=1}^4 \int dr_j \right) U(r_1 - r_2) U(r_3 - r_4) \int d\Omega \Omega$$

$$\times \left\{ \coth \frac{\Omega}{2T} - \tanh \frac{\Omega}{2T} \right\} \mathcal{K}_2(\{r_j\}, 0, 0, \varepsilon' \sim T, \Omega)$$

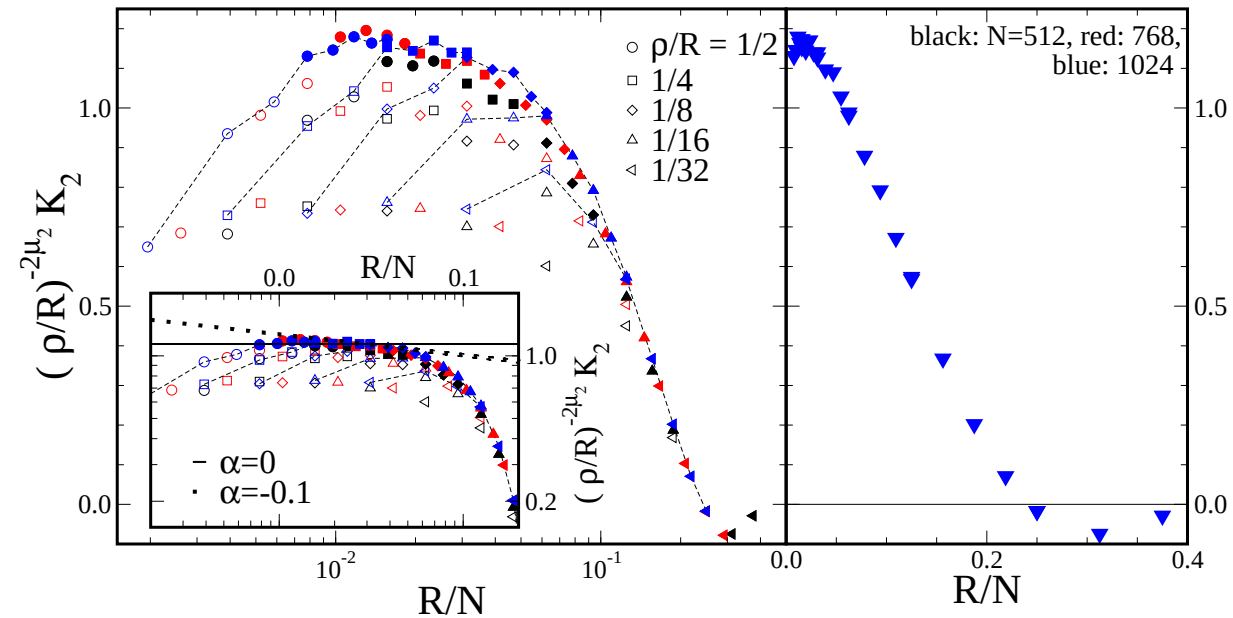
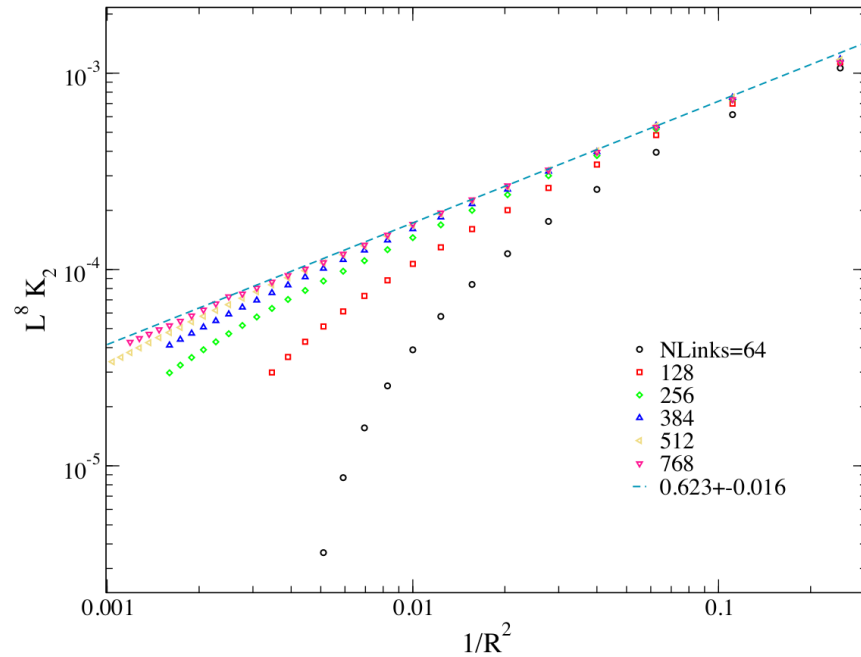
$$\mathcal{K}_2(\{r_j\}, E, \varepsilon, \varepsilon', \Omega) = \frac{\Delta^4}{8} \left\langle \sum_{\alpha\beta\gamma\delta} \mathcal{B}_{\alpha\beta}^*(r_1, r_2) \mathcal{B}_{\delta\gamma}(r_1, r_2) \mathcal{B}_{\gamma\delta}^*(r_3, r_4) \mathcal{B}_{\beta\alpha}(r_3, r_4) \right.$$

$$\left. \times \delta(E - \epsilon_\alpha) \delta(\varepsilon' + \Omega - \epsilon_\beta) \delta(\varepsilon' - \epsilon_\gamma) \delta(\varepsilon + \Omega - \epsilon_\delta) \right\rangle.$$

$$\mathcal{K}_2(\{r_j\}, 0, 0, \varepsilon' \sim \Omega, \Omega) = L^{-4d} \left(\frac{|r_1 - r_2|}{R} \frac{|r_3 - r_4|}{R} \right)^{\mu_2} \left(\frac{R}{L_\Omega} \right)^\alpha$$

$$R = (r_1 + r_2 - r_3 - r_4)/2$$

Interaction scaling at quantum Hall critical point: Second order



$\mu_2 = 0.62 \pm 0.05$ in agreement with scaling of first order

$\alpha = -0.05 \pm 0.1$ (in fact, exactly zero for unitary class; see below)

Exponent α drops out of the expression for τ_ϕ^{-1}

if $\alpha > 2\mu_2 - d$ — fulfilled for QH transition

Scaling at QH transition: Theory and experiment

- Theory (short-range interaction):

→ dephasing rate $\tau_{\phi}^{-1} \propto T^p$ with $p = 1 + 2\mu_2/d$

dephasing length $L_{\phi} \propto T^{-1/z_T}$ $z_T = d/p$

Transition width exponent $\kappa = \frac{1}{z_T \nu} = \frac{1 + 2\mu_2/d}{\nu d}$

$\mu_2 \simeq 0.62 \longrightarrow p \simeq 1.62 \longrightarrow z_T \simeq 1.23$

$\nu \simeq 2.35$ (Huckestein et al '92, ...) $\longrightarrow \kappa \simeq 0.346$

$\nu \simeq 2.59$ (Ohtsuki, Slevin '09) $\longrightarrow \kappa \simeq 0.314$

- Experiment (long-range $1/r$ Coulomb interaction):

$\kappa = 0.42 \pm 0.01$

Difference in κ fully consistent with short-range and Coulomb ($1/r$) problems being in different universality classes

Anderson transition: $2 + \epsilon$ dimensions, short-range interaction

$$-dt/d \ln L \equiv \beta(t) = \epsilon t - 2t^3 - 6t^5 + O(t^7) \quad (\text{Wegner '89})$$

$$t = 1/2\pi g \quad g - \text{dimensionless conductance}$$

Metal-insulator transition at
$$t_* = \left(\frac{\epsilon}{2}\right)^{1/2} - \frac{3}{2} \left(\frac{\epsilon}{2}\right)^{3/2} + O(\epsilon^{5/2})$$

Localization length index
$$\nu = -1/\beta'(t_*) = \frac{1}{2\epsilon} - \frac{3}{4} + O(\epsilon)$$

Exponents controlling scaling of interaction:

$$\mu_2 = \sqrt{2}\epsilon - \frac{3}{2}\zeta(3)\epsilon^2 + O(\epsilon^{5/2}) \quad \alpha = O(\epsilon^{5/2})$$

Temperature scaling of transition:

$$z_T = 2 - 2\sqrt{2}\epsilon^{1/2} + 5\epsilon - 4\sqrt{2}\epsilon^{3/2} + O(\epsilon^2)$$

$$\kappa = \epsilon + \sqrt{2}\epsilon^{3/2} + \epsilon^2 + \epsilon^{5/2}/\sqrt{2} + O(\epsilon^3)$$

Anderson transition: $2 + \epsilon$ dimensions, Coulomb interaction

Broken time-reversal symmetry (unitary class), 2-loop calculation

Baranov, Burmistrov, Pruisken, PRB '02

$$\beta(t) = \epsilon t - 2t^2 - 4At^3 ; \quad A \simeq 1.64$$

$$t_* = \frac{\epsilon}{2} - \frac{A}{2}\epsilon^2 + O(\epsilon^3)$$

$$\nu = \frac{1}{\epsilon} - A + O(\epsilon)$$

$$z = z_T = 2 + \frac{\epsilon}{2} + \left(\frac{A}{2} - \frac{\pi^2}{24} - \frac{3}{4} \right) \epsilon^2 + O(\epsilon^3)$$

$$\kappa = \frac{\epsilon}{2} + \left(\frac{A}{2} - \frac{1}{8} \right) \epsilon^2 + O(\epsilon^3)$$

Exponents for short-range and Coulomb interaction are different!

Anderson transition in 3D: Short-range vs Coulomb

Theory, short-range interaction:

$$\nu = 1.57 \pm 0.02 \quad (\text{Slevin, Ohtsuki '99})$$

μ_2 and α remain to be calculated $\longrightarrow z_T, \kappa$

Coulomb interaction: no controllable theory for exponents

Experiment (Coulomb):

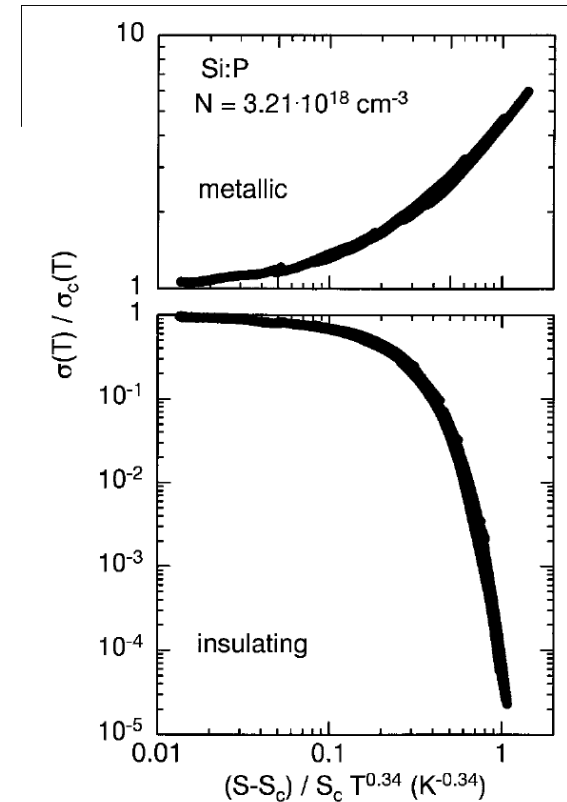
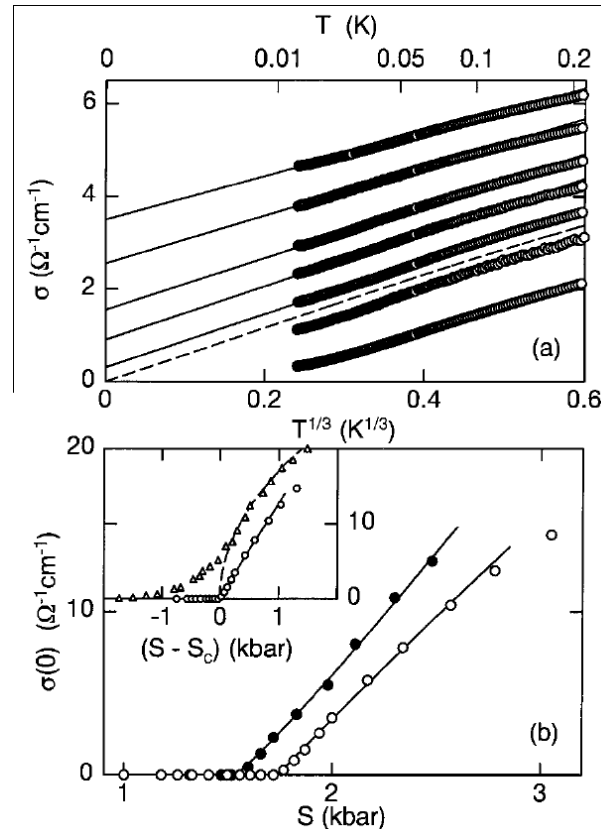
$$s = \nu \simeq 1.0 \pm 0.1$$

z_T in the range from 2 to 3

Experiment, short-range:

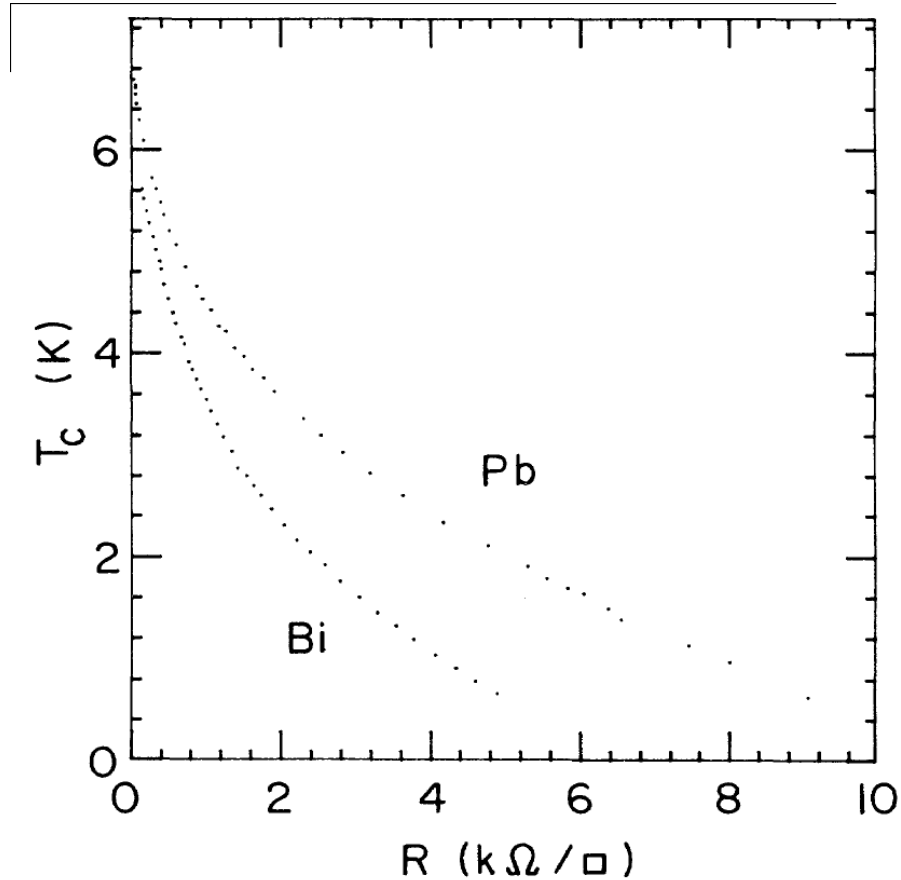
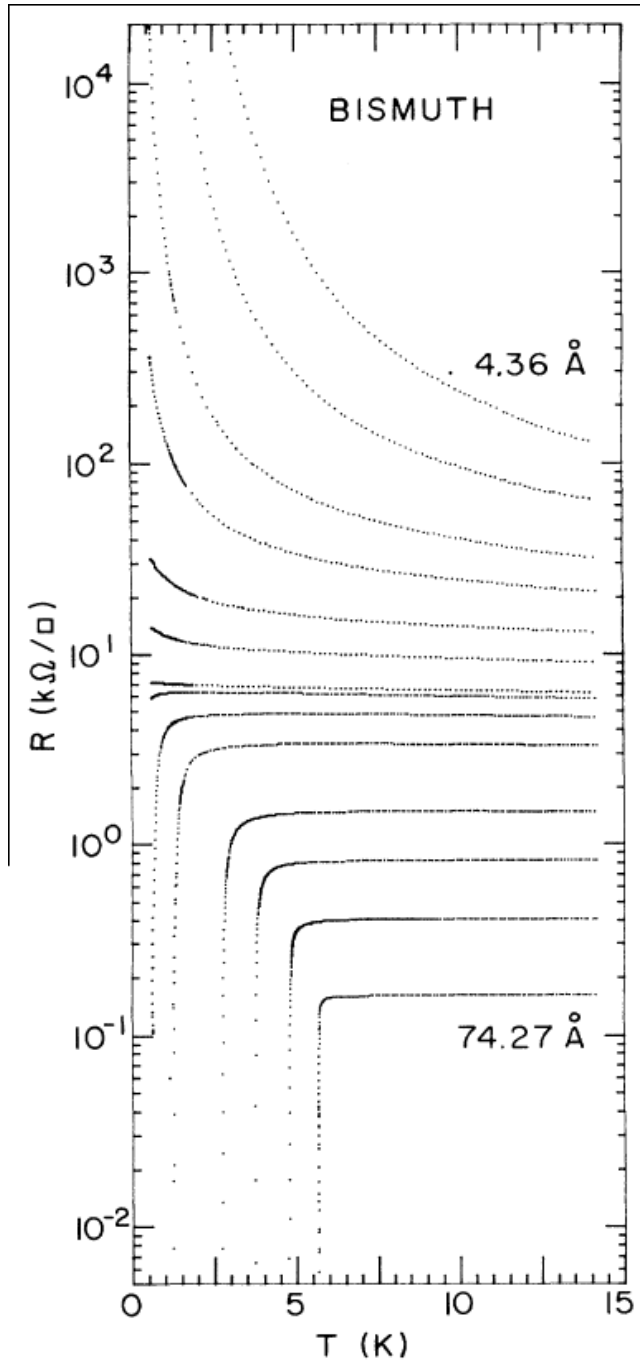
not available

Cold atom systems?



Waffenschmidt, Pfeleiderer, v. Löhneysen, PRL'99

Superconductor-Insulator Transition



Haviland, Liu, Goldman, PRL'89

Bi and Pb films

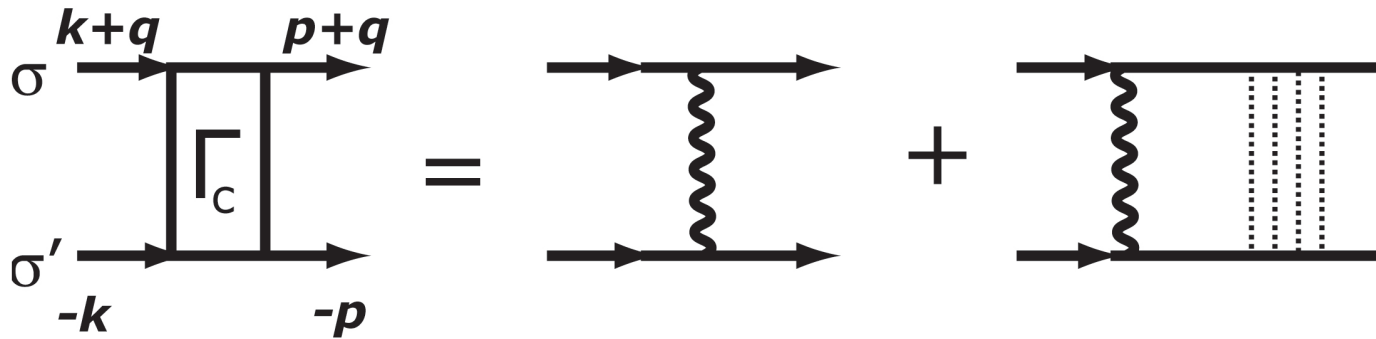
Suppression of T_c by disorder

Anderson theorem

Abrikosov, Gorkov'59 ; Anderson'59

non-magnetic impurities do not affect s-wave superconductivity:

Cooper instability unaffected by diffusive motion



mean free path does not enter the expression for T_c



Anderson Theorem vs Anderson Localization – ?

Suppression of T_c of disordered films due to Coulomb repulsion

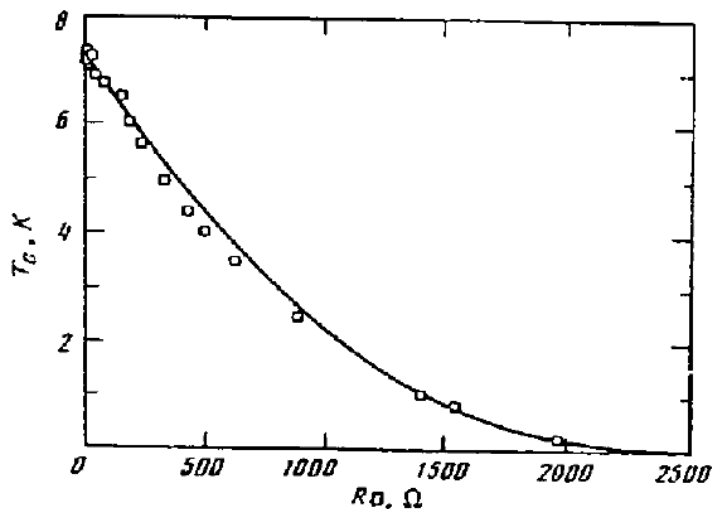
Combined effect of disorder and Coulomb (long-range) interaction

First-order perturbative correction to T_c : Maekawa, Fukuyama'81

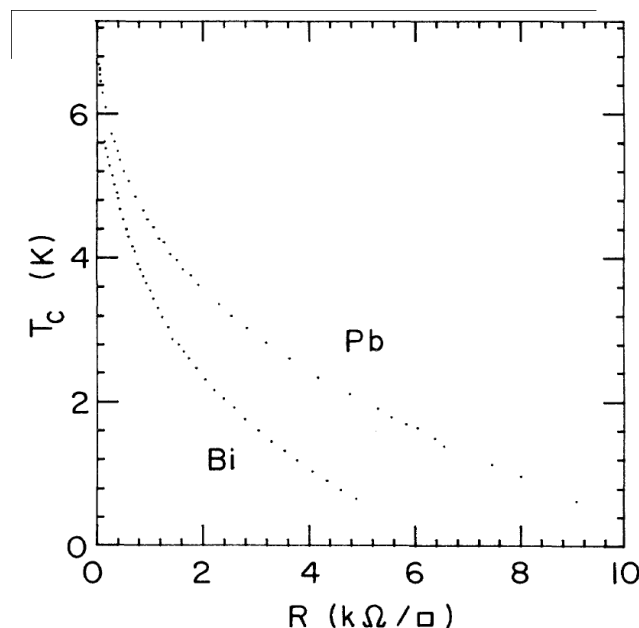
RG theory: Finkelstein '87

T_c suppressed; monotonously decays with increasing resistivity

This suppression is observed in many experiments



Mo-Ge films, Graybeal, Besley'84



Bi and Pb films, Haviland, Liu, Goldman'89

Enhancement of superconductivity by multifractality

short-range interaction

Feigelman, Ioffe, Kravtsov, Yuzbashyan, Cuevas, PRL '07, Ann. Phys.'10 :

multifractality of wave functions near MIT in 3D

→ enhancement of Cooper-interaction matrix elements

→ enhancement of T_c as given by self-consistency equation

Questions:

- Can suppression of T_c for Coulomb repulsion and enhancement due to multifractality be described in a unified way?
- What are predictions of RG ? Does the enhancement hold if the repulsion in particle-hole channels is taken into account ?
- Effect of disorder on T_c in 2D systems ?

SIT in disordered 2D system: Orthogonal symmetry class
 σ -model RG with short-range interaction:

$$\frac{dt}{dy} = t^2 - \left(\frac{\gamma_s}{2} + 3\frac{\gamma_t}{2} + \gamma_c\right)t^2$$

$$\frac{d\gamma_s}{dy} = -\frac{t}{2}(\gamma_s + 3\gamma_t + 2\gamma_c)$$

$$\frac{d\gamma_t}{dy} = -\frac{t}{2}(\gamma_s - \gamma_t - 2\gamma_c)$$

$$\frac{d\gamma_c}{dy} = -\frac{t}{2}(\gamma_s - 3\gamma_t) - 2\gamma_c^2 \quad y \equiv \ln L$$

Interactions: **singlet** γ_s , **triplet** γ_t , **Cooper** γ_c

$\gamma_s \rightarrow -1 \quad \longrightarrow \quad$ **Finkelstein's RG for Coulomb interaction**

Disorder: dimensionless **resistivity** $t = 1/G$

Assume small bare values: $t_0, \gamma_{i,0} \ll 1$

SIT in disordered 2D system: Orthogonal class (cont'd)

Weak interaction $\longrightarrow$ discard $\gamma_i t^2$ contributions to $dt/d \ln L$

$$\frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} = -\frac{t}{2} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2\gamma_c^2 \end{pmatrix} ; \quad \frac{dt}{dy} = t^2$$

Eigenvalues and -vectors of linear problem (without BCS term γ_c^2):

$$\lambda = 2t : \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} ; \quad \lambda' = -t : \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

2D system is “weakly critical” (on scales shorter than ξ)

The **eigenvalues** λ , λ' are exactly **multifractal exponents**:

$\lambda \equiv -\Delta_2 > 0$ (RG relevant), $\lambda' = -\mu_2 < 0$ (RG irrelevant)

SIT in disordered 2D system: Orthogonal class (cont'd)

Couplings that diagonalize the linear system:

$$\begin{pmatrix} \gamma \\ \gamma' \\ \gamma'' \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix}$$

Upon RG γ increases, whereas γ' , γ'' decrease.

Solution approaches the λ -eigenvector, i.e.. $\gamma_s = -\gamma_t = -\gamma_c$

→ neglect γ' , γ'' and keep γ only:

$$\frac{d\gamma}{dy} = 2t\gamma - \frac{2}{3}\gamma^2 \qquad t(y) = \frac{t_0}{1 - t_0 y}$$

Superconductivity may develop if the starting value

$$\gamma_0 = \frac{1}{6}(-\gamma_{s,0} + 3\gamma_{t,0} + 2\gamma_{c,0}) < 0$$

SIT in disordered 2D systems, orthogonal class: Results

$$T_c \sim \exp \{ -1/|\gamma_{c,0}| \} \quad (\text{BCS}) , \quad G_0 \gtrsim |\gamma_0|^{-1}$$

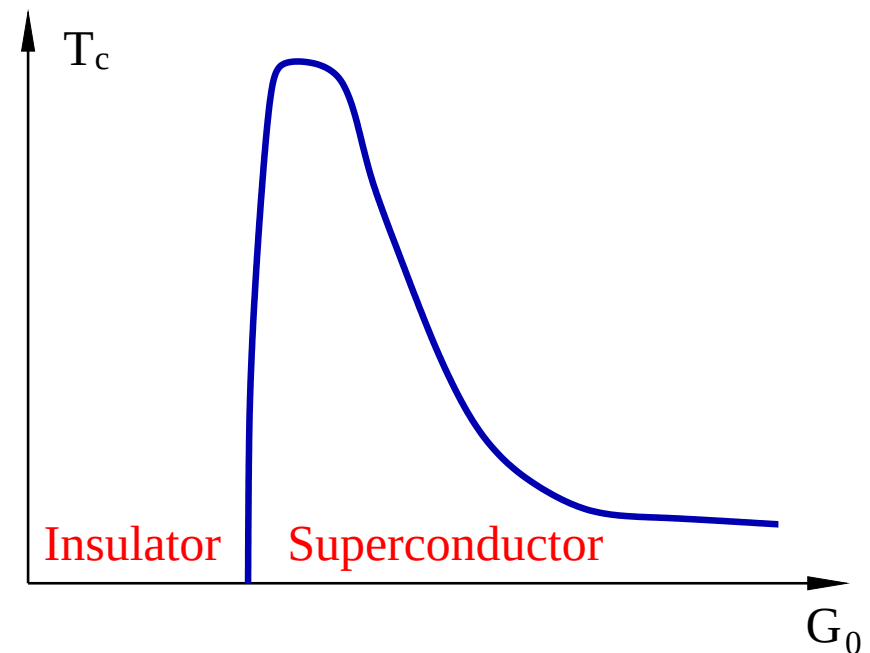
$$T_c \sim \exp \{ -2G_0 \} , \quad |\gamma_0|^{-1/2} \lesssim G_0 \lesssim |\gamma_0|^{-1}$$

$$\text{insulator} , \quad G_0 \lesssim |\gamma_0|^{-1/2}$$

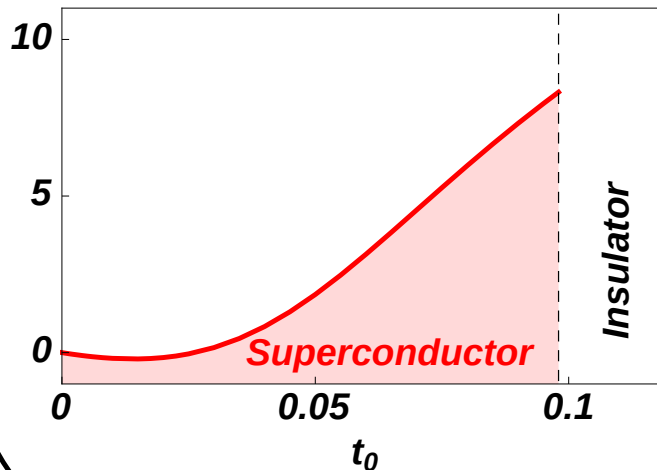
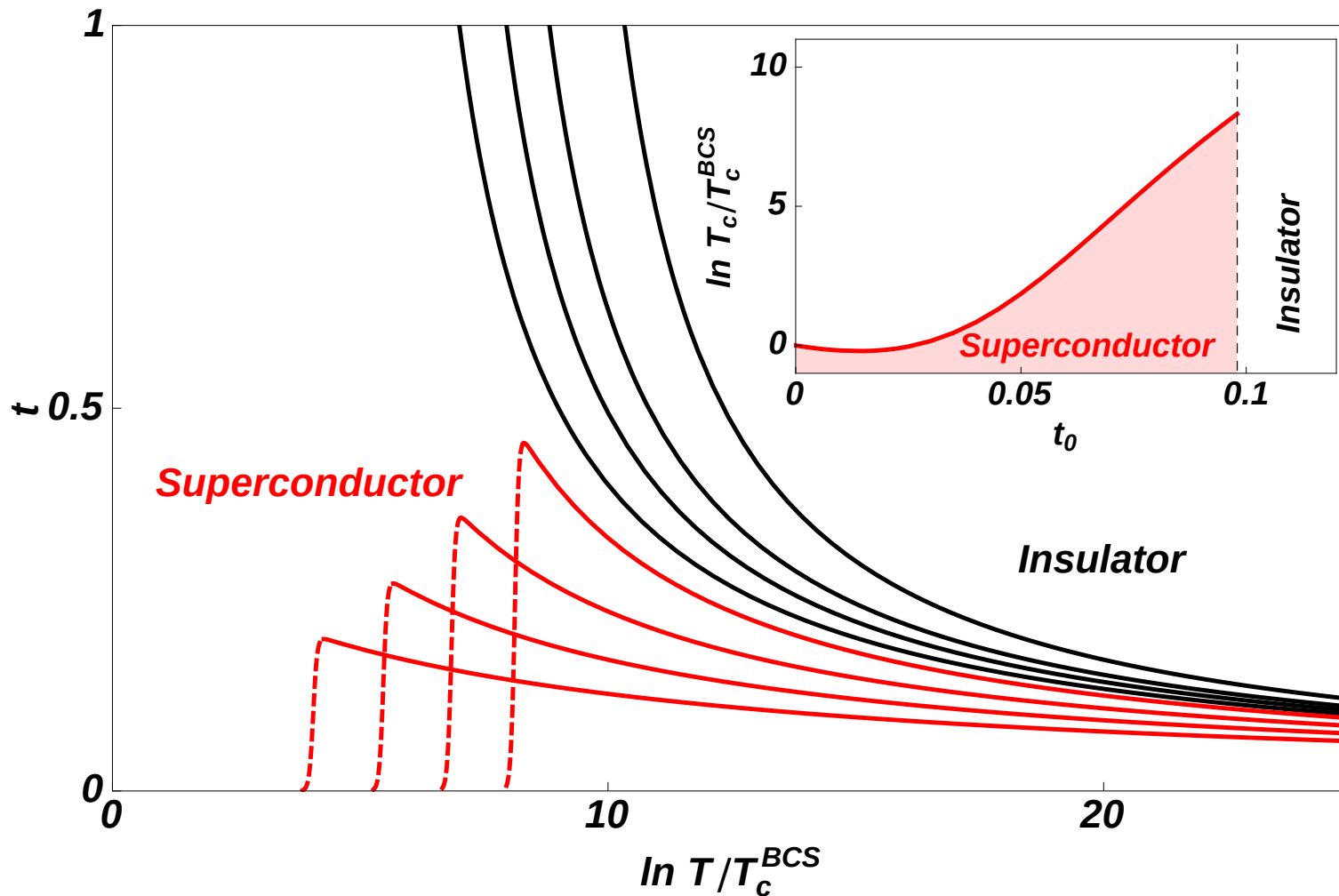
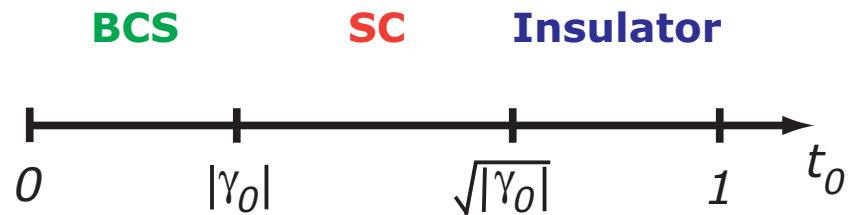
Non-monotonous dependence
of T_c on disorder (G_0)

Exponentially strong enhancement
of superconductivity by multifractality
in the intermediate disorder range,

$$|\gamma_0|^{-1/2} \lesssim G_0 \lesssim |\gamma_0|^{-1}$$



SIT in disordered 2D system, orth. class: Results (cont'd)



Inset: $T_c(t_0)$

$t(T)$ for
 $\gamma_{c0} = 0.04$,
 $\gamma_{s0} = -0.005$,
 $\gamma_{t0} = 0.005$,
 and
 $t_0 = 0.065 \div 0.12$

SIT near Anderson transition

Consider system at Anderson localization transition
in 2D (symplectic symmetry class) or 3D

$$\frac{d\gamma}{dy} = -\Delta_2\gamma - \gamma^2$$

Superconductivity if $\gamma_0 < 0$

$\Delta_2 < 0$ – **multifractal exponent** at Anderson transition point

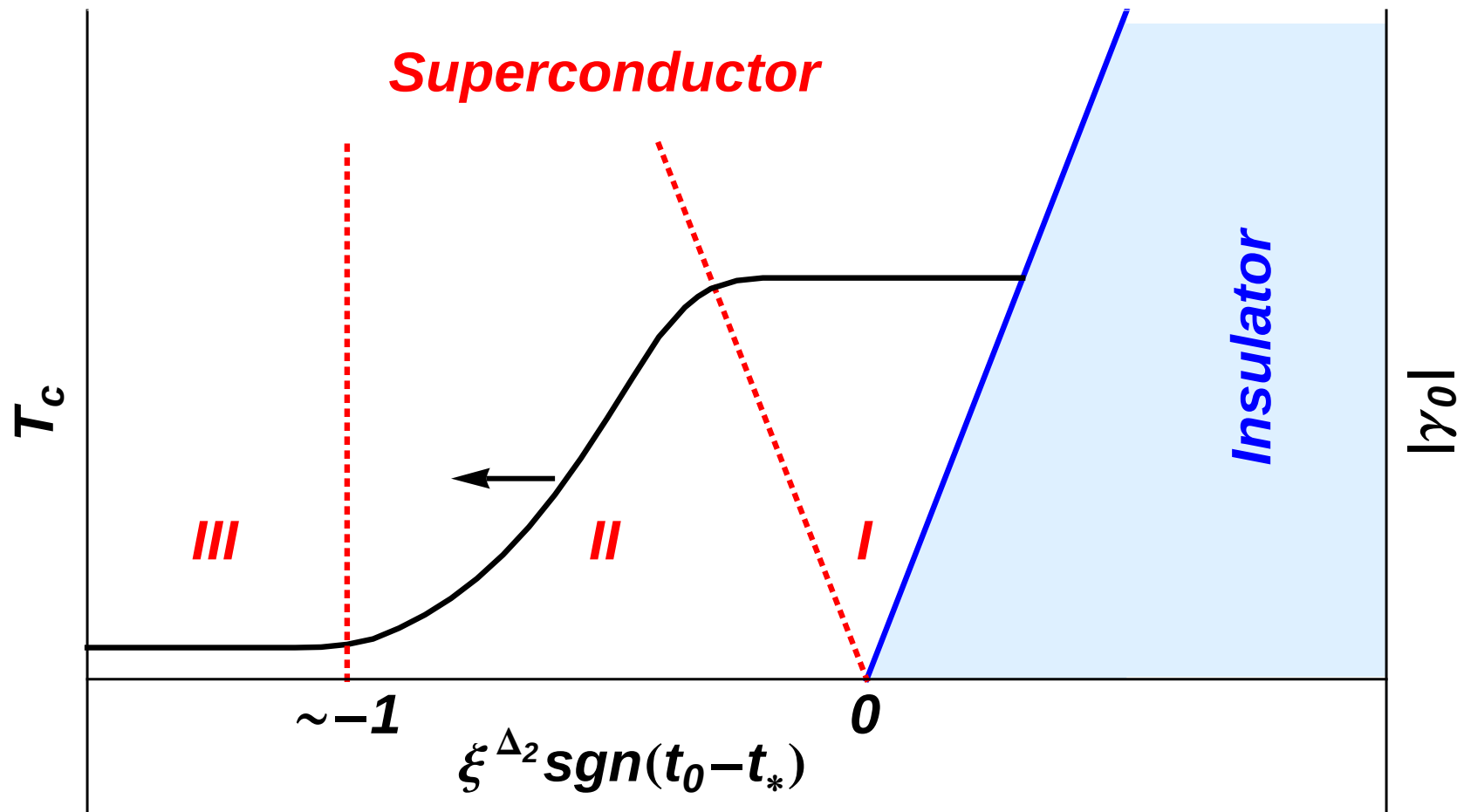
$$T_c \sim |\gamma_0|^{d/|\Delta_2|}$$

Exponentially strong enhancement of superconductivity:

Power-law instead of exponential dependence of T_c on interaction!

Agrees with **Feigelman et al.**

SIT near Anderson transition: Results



III : BCS

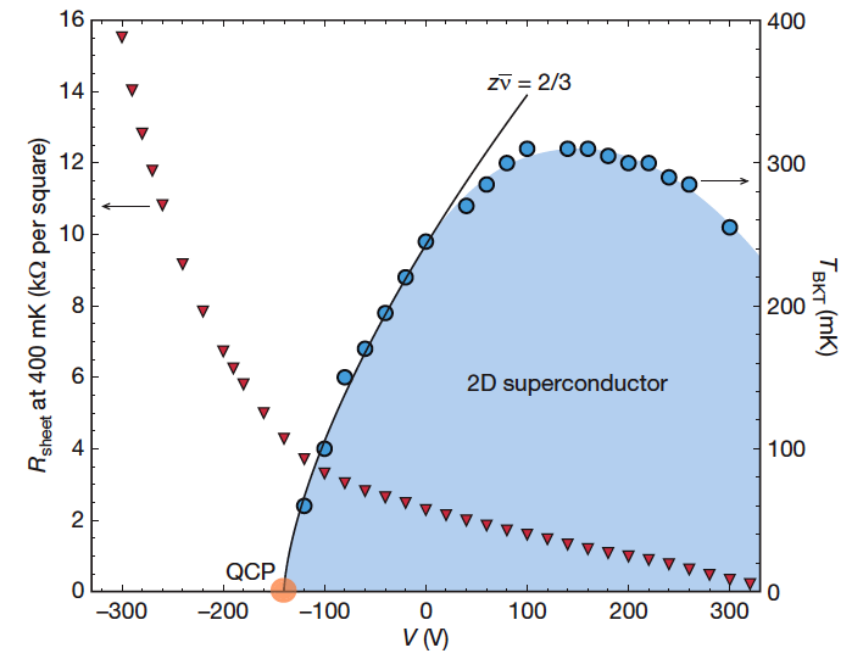
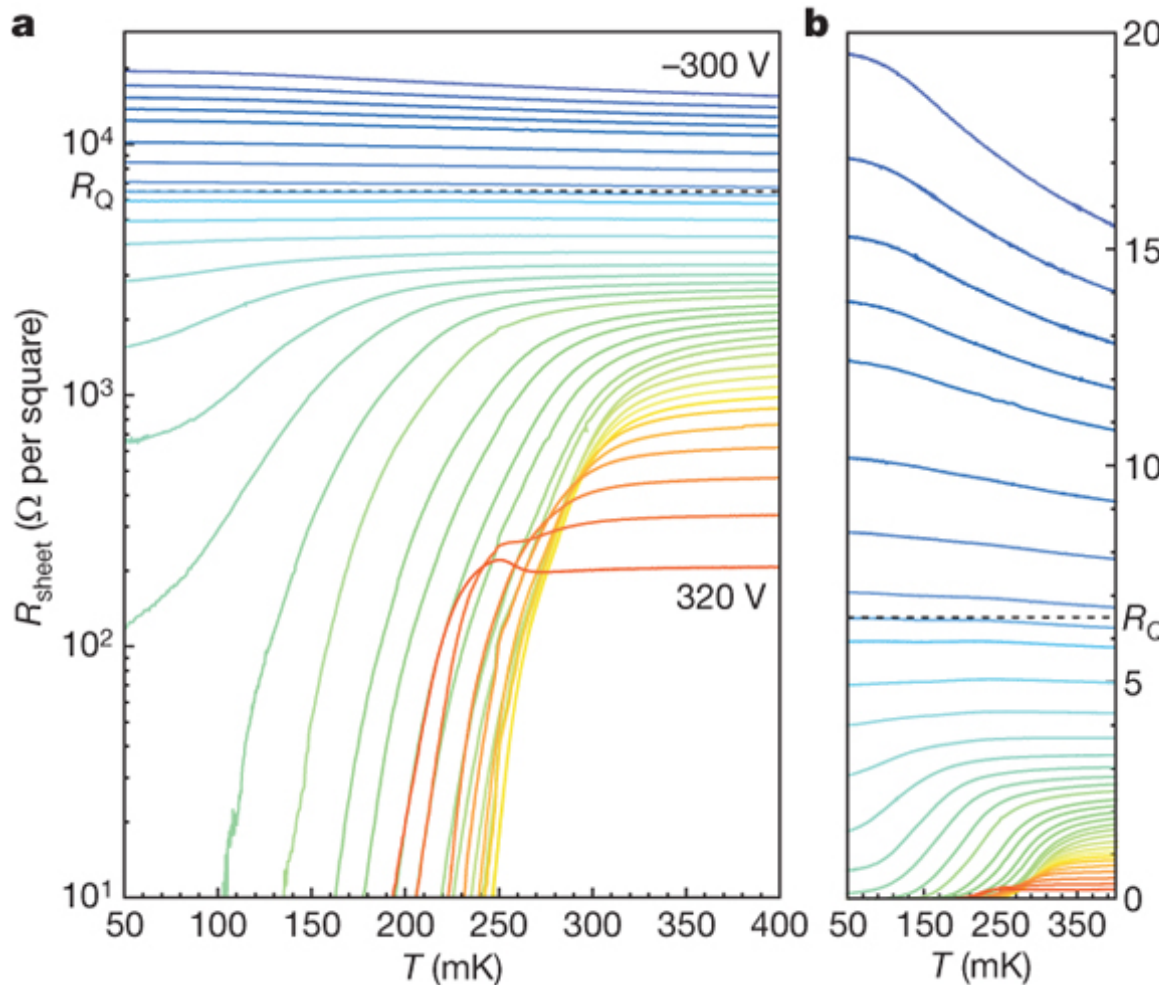
I : $T_c \sim |\gamma_0|^{d/|\Delta_2|}$

II : crossover: $T_c \sim \xi^{-3} \exp(-c\xi^{\Delta_2}/|\gamma_0|)$ (3D)

Experimental realizations ?

Key assumption: short-range character of interaction

→ systems with strongly screened Coulomb interaction



Caviglia, . . . , Mannhart,
Triscone, Nature'08

$\text{LaAlO}_3/\text{SrTiO}_3$ interface

$\epsilon \approx 10^4$

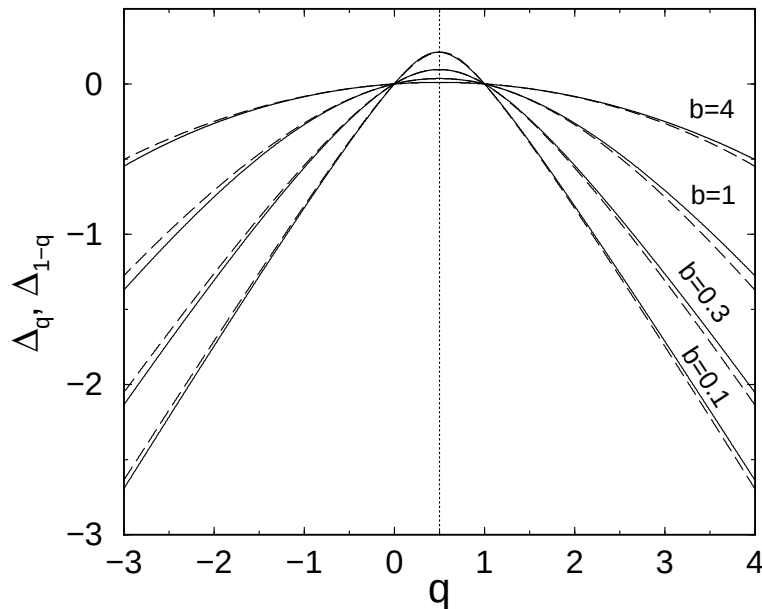
Symmetry of multifractal spectra

ADM, Fyodorov, Mildenberger, Evers '06

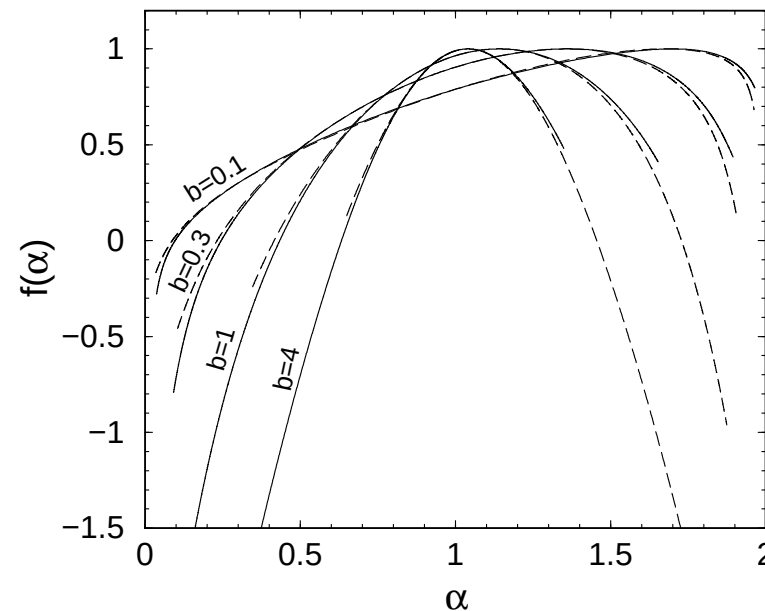
LDOS distribution in σ -model + universality

→ exact symmetry of the multifractal spectrum:

$$\Delta_q = \Delta_{1-q}$$



$$f(2d - \alpha) = f(\alpha) + d - \alpha$$



→ probabilities of unusually large
and unusually small $|\psi^2(r)|$ are related !

Symmetries of multifractal spectra (cont'd)

- Relation to invariance of the σ model correlation functions with respect to **Weyl group** of the σ model target space; generalization to **unconventional symmetry classes**

Gruzberg, Ludwig, ADM, Zirnbauer PRL'11

- generalization on **full set of composite operators**, i.e. also on subleading ones.

Gruzberg, ADM, Zirnbauer, in preparation

Classification of scaling observables

Consider n points $\mathbf{r}_1, \dots, \mathbf{r}_n$ and n wave functions $\psi_1, \dots, \psi_n$.

For each $p \leq n$ define

$$A_p(\mathbf{r}_1, \dots, \mathbf{r}_{\tilde{p}}) = |D_p(\mathbf{r}_1, \dots, \mathbf{r}_p)|^2$$

$$D_p(\mathbf{r}_1, \dots, \mathbf{r}_p) = \text{Det} \begin{pmatrix} \psi_1(\mathbf{r}_1) & \cdots & \psi_1(\mathbf{r}_p) \\ \vdots & \ddots & \vdots \\ \psi_p(\mathbf{r}_1) & \cdots & \psi_p(\mathbf{r}_p) \end{pmatrix}$$

For any set of complex $q_1, \dots, q_n$ define

$$K_{(q_1, \dots, q_n)} = \langle A_1^{q_1 - q_2} A_2^{q_2 - q_3} \cdots A_{n-1}^{q_{n-1} - q_n} A_n^{q_n} \rangle .$$

These are pure-scaling correlators of wave functions.

The proof goes via a mapping to the sigma model.

Scaling operators in sigma-model formalism

Sigma-model composite operators

corresponding to wave function correlators $K_{(q_1, \dots, q_n)}$ are

$$\mathcal{O}_{(q_1, \dots, q_n)}(Q) = d_1^{q_1 - q_2} d_2^{q_2 - q_3} \dots d_n^{q_n},$$

where d_j is the principal minor of size $j \times j$ of the matrix (block of Q in retarded-advanced and boson-fermion spaces)

$$(1/2)(Q_{11} - Q_{22} + Q_{12} - Q_{21})_{bb}.$$

These are pure scaling operators. Two alternative proofs:

- Iwasawa decomposition $G = NAK$.

Functions $\mathcal{O}_{(q_1, \dots, q_n)}(Q)$ are N -invariant spherical functions on G/K and have a form of “plane waves” on A

- $\mathcal{O}_{(q_1, \dots, q_n)}(Q)$ as highest-weight vectors

Iwasawa decomposition

σ -model space: G/K K — maximal compact subgroup

consider for definiteness unitary class (e.g., QH transition)

$$G/K = \mathrm{U}(n, n|2n)/[\mathrm{U}(n|n) \times \mathrm{U}(n|n)]$$

Iwasawa decomposition: $G = NAK$ $g = nak$

A — maximal abelian in G/K

N — nilpotent

($\longleftrightarrow$ triangular matrices with 1 on the diagonal)

Particular example:

Gram decomposition: matrix = triangular $\times$ unitary

Spherical functions

Eigenfunctions of G -invariant operators (like RG transformation) are spherical functions on G/K .

N -invariant spherical functions on G/K are “plane waves”

$$\varphi_{q,p} = \exp \left(-2 \sum_{j=1}^n q_j x_j - 2i \sum_{l=1}^n p_l y_l \right)$$

$x_1, \dots, x_n; y_1, \dots, y_n$ — natural coordinates on abelian group A .

Here q_j can be arbitrary complex, p_j are non-negative integers.

For $p_j = 0$

the function ϕ_q is exactly $\mathcal{O}_{(q_1, \dots, q_n)}(Q)$ introduced above

Symmetries of scaling exponents

Weyl group $\longrightarrow$ invariance of eigenvalues
of any G invariant operator with respect to

(i) reflections

$$q_j \rightarrow -c_j - q_j \qquad c_j = 1 - 2j$$

(ii) permutations

$$q_i \rightarrow q_j + \frac{c_j - c_i}{2}; \qquad q_j \rightarrow q_i + \frac{c_i - c_j}{2}$$

This is valid in particular for eigenvalues of RG,
i.e. scaling exponents

Symmetries of multifractal spectrum of A_2

$$A_2 = V^2 |\psi_1(r_1)\psi_2(r_2) - \psi_1(r_2)\psi_2(r_1)|^2$$

$\longleftrightarrow$ Hartree-Fock matrix element of e-e interaction

scaling: $\langle A_2^q \rangle \propto L^{-\Delta_{q,q}}$

symmetry: $\Delta_{q,q} = \Delta_{2-q,2-q}$

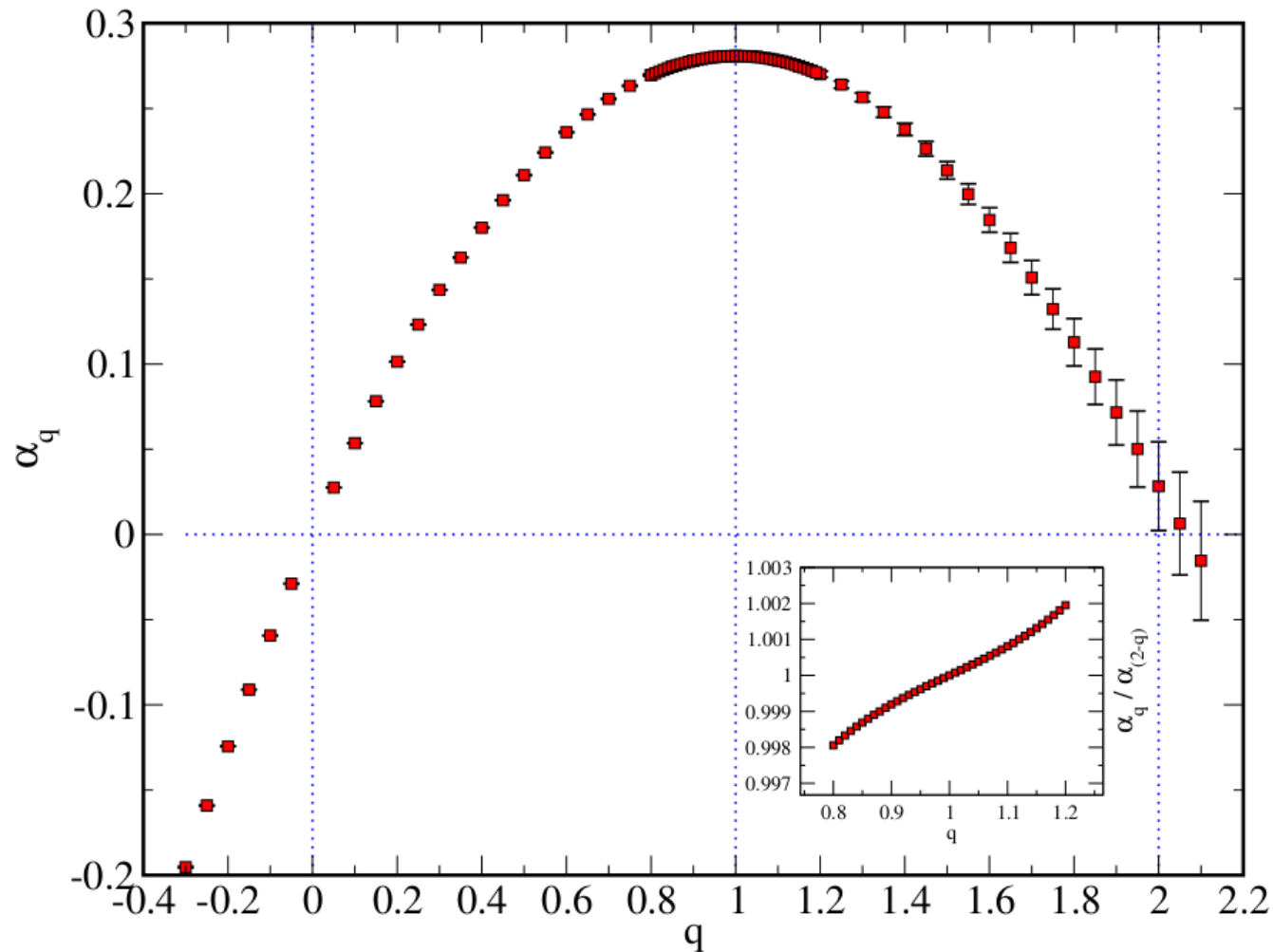
Relation to operators introduced above

(dephasing at QH and MI transitions):

$\mu_2 \equiv \Delta_{1,1}$ $\alpha \equiv \Delta_{2,2}$ Symmetry $\longrightarrow \Delta_{2,2} = \Delta_0 = 0$

Multifractal spectrum of A_2 at quantum Hall transition

Numerical data: Bera, Evers, unpublished



Confirms the symmetry $q \longleftrightarrow 2 - q$

Summary

- Multifractality of wave functions – remarkable property of Anderson localization transitions
- σ model RG: systematic, controllable theoretical description
- Multifractality strongly affects interaction-induced physics in problems with short-range interaction
- Multifractality determines scaling of dephasing rate and transition width at MI and QH transitions
- Non-monotonous dependence of T_c on resistivity; exponential enhancement of superconductivity by multifractality in 2D systems and near Anderson transition
- Classification of operators describing wave function correlations
- Symmetries of scaling exponents: Weyl group invariance