

# A quantum dot close to Stoner instability: the role of Berry's phase

Alexander Shnirman (KIT, Karlsruhe)

A. Saha & Y. Gefen (Weizmann)

A. Altland (Cologne)

M. Kiselev (ICTP, Trieste)

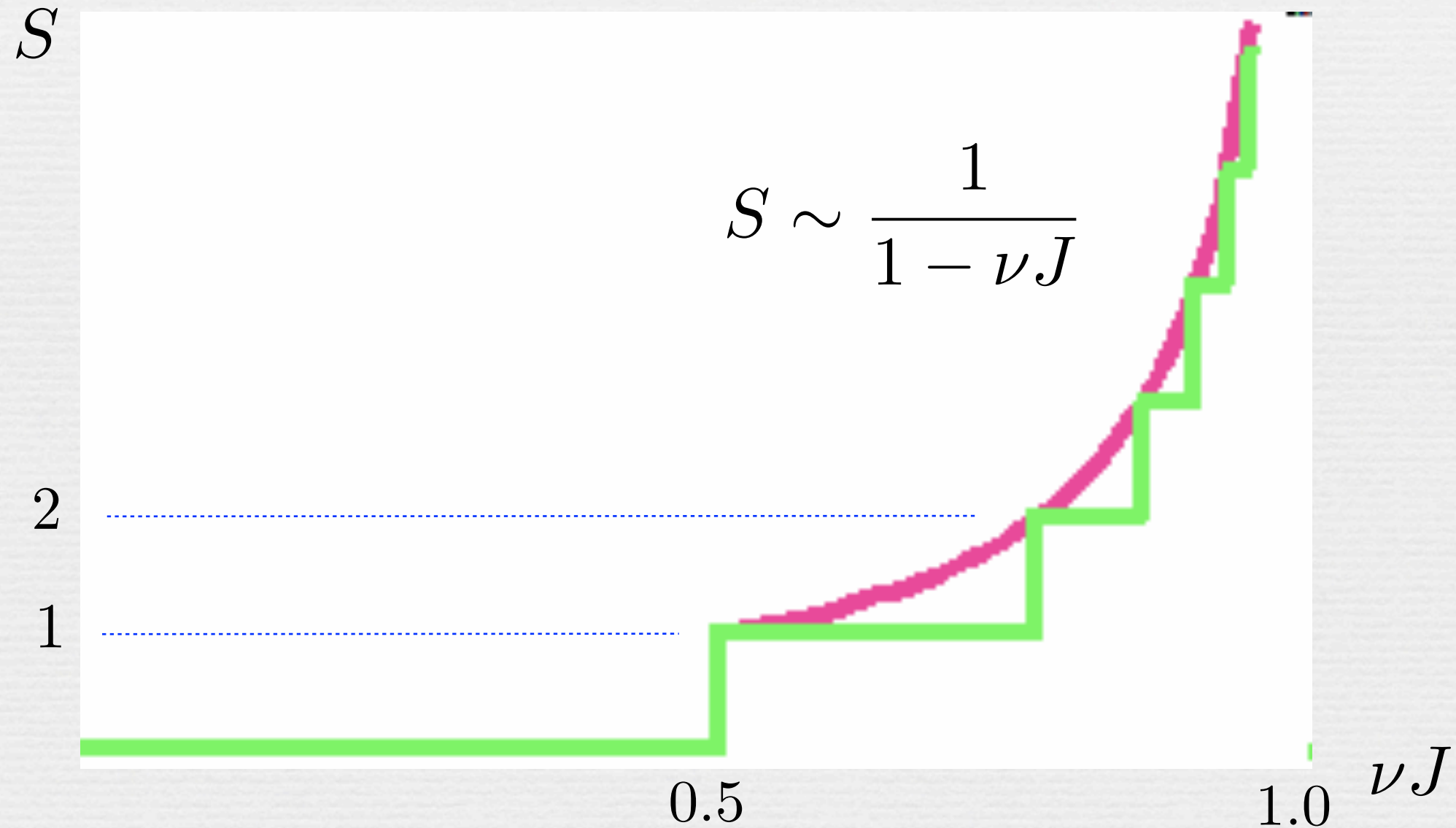
I. Burmistrov (Landau)

# Outline

- Motivation
- Isolated dot - susceptibility **A. Saha et al.  
Annals of Phys., 327 (10), 2543 (2012)**
- Open dot - AES action (unfinished)
- Conclusions

# Motivation: Mesoscopic Stoner Instability

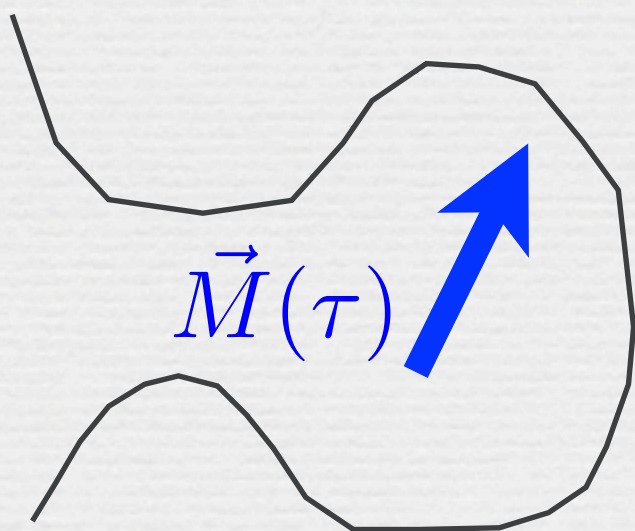
$$H = \sum_{\alpha, \sigma} \epsilon_{\alpha} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} - J \hat{\mathbf{S}}^2 \quad \langle \hat{\mathbf{S}}^2 \rangle = S(S + 1)$$



## Motivation:

- Dots close to Stoner regime
- Small ferromagnetic dots (super-paramagnets)
- 1-D and 2-D arrays of dots

Vision: description in terms of slow  
collective variable: **magnetization**  $\vec{M}(\tau)$



LLG equation

$$\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B} - \eta \vec{M} \times \frac{d\vec{M}}{dt}$$

# Universal Hamiltonian

A. V. Andreev, A. Kamenev, Phys. Rev. Lett. 81, 3199 (1998)

I.L. Kurland, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B 62, 14886 (2000)

$$H = H_0 + H_C + H_J + H_\lambda$$

$$H_0 = \sum_{\alpha, \sigma} \epsilon_\alpha a_{\alpha, \sigma}^\dagger a_{\alpha, \sigma}$$

Coulomb

$$H_C = E_C \left( \hat{N} - N_0 \right)^2 \quad \hat{N} = \sum_{\alpha, \sigma} a_{\alpha, \sigma}^\dagger a_{\alpha, \sigma}$$

Exchange

$$H_J = -J \hat{\mathbf{S}}^2 \quad \hat{\mathbf{S}} = \sum_{\alpha} a_{\alpha, \sigma_1}^\dagger \mathbf{S}_{\sigma_1 \sigma_2} a_{\alpha, \sigma_2}$$

Cooper

$$H_\lambda = \lambda T^\dagger T, \quad T = \sum_{\alpha} a_{\alpha, \uparrow} a_{\alpha, \downarrow}$$

# Coulomb part

A. Kamenev, Y. Gefen, Phys. Rev. B 54, 5428 (1996)

$$H = \sum_{\alpha, \sigma} \epsilon_{\alpha} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} + E_C (\hat{N} - N_0)^2 \quad \mathcal{Z} = \int D\bar{\Psi} D\Psi e^{\mathcal{S}}$$

## Hubbard-Stratonovich

$$\mathcal{S}_{\Psi, \phi} = \int_0^{\beta} d\tau \left[ \sum_{\alpha} \bar{\Psi}_{\alpha} [-\partial_{\tau} - \epsilon_{\alpha} + \mu - i\phi(\tau)] \Psi_{\alpha} - \frac{\phi^2(\tau)}{4E_C} + i\phi(\tau)N_0 \right]$$

$$\mathcal{S}_{\phi} = \sum_{\alpha} \text{tr} \ln (-\partial_{\tau} - \epsilon_{\alpha} + \mu - i\phi) - \int_0^{\beta} d\tau \left[ \frac{\phi^2}{4E_C} - i\phi N_0 \right]$$

# Coulomb part

A. Kamenev, Y. Gefen, Phys. Rev. B 54, 5428 (1996)

$$\mathcal{S}_\alpha = \text{tr} \ln [-\partial_\tau - \epsilon_\alpha + \mu - i\phi]$$

$$\mathcal{S}_\alpha = \text{tr} \ln [R^{-1} \{-\partial_\tau - \epsilon_\alpha + \mu - i\phi\} R]$$

$$R \in U(1) \quad \begin{aligned} \Psi(\tau) &\rightarrow R(\tau)\Psi(\tau) \\ R(\tau + \beta) &= R(\tau) \end{aligned}$$

$$\mathcal{S}_\alpha = \text{tr} \ln [-\partial_\tau - \epsilon_\alpha + \mu - i\phi - R^{-1} \partial_\tau R]$$

$$i\phi(\tau) + R^{-1} \partial_\tau R \rightarrow \phi_0$$

$$-\pi/\beta < \phi_0 < \pi/\beta$$

Zero mode

Winding numbers

$$\mathcal{Z}(\mu) \propto \int d\phi_0 Z_{\text{free}}(\mu - i\phi_0) e^{-\frac{\beta \phi_0^2}{4E_C} + i\beta \phi_0 N_0}$$

# Exchange part

$$H = \sum_{\alpha, \sigma} \epsilon_{\alpha} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} - J \hat{\mathbf{S}}^2$$

$$\mathcal{S}_{\Psi, \Phi} = \int_0^{\beta} d\tau \left[ \sum_{\alpha} \bar{\Psi}_{\alpha} \left( -\partial_{\tau} - \epsilon_{\alpha} + \mu - \vec{\Phi} \cdot \vec{\mathbf{S}} \right) \Psi_{\alpha} - \frac{|\vec{\Phi}|^2}{4J} \right]$$

$$\mathcal{S}_{\Phi} = \sum_{\alpha} \text{tr} \ln \left( -\partial_{\tau} - \epsilon_{\alpha} + \mu - \vec{\Phi} \cdot \vec{\mathbf{S}} \right) - \int_0^{\beta} d\tau \frac{|\vec{\Phi}|^2}{4J}$$

**Non-Abelian**



M.N.Kiselev, Y.Gefen, Phys. Rev. Lett. 96, 066805 (2006)

I.Burmistrov, Y.Gefen, M.Kiselev, Pis'ma v ZhETF 92, 202 (2010)

# Rotation

$$\mathcal{S}_\Phi = \sum_\alpha \text{tr} \ln \left( -\partial_\tau - \epsilon_\alpha + \mu - \Phi(\tau) \vec{n}(\tau) \cdot \frac{\vec{\sigma}}{2} \right) - \int_0^\beta d\tau \frac{\Phi^2(\tau)}{4J}$$

$$\Phi(\tau) \equiv |\vec{\Phi}(\tau)| \qquad \vec{n} \cdot \vec{\sigma} = R \sigma_z R^{-1}$$

$$R \in SU(2)/U(1)$$

$$\Psi(\tau) \rightarrow R(\tau) \Psi(\tau)$$

$$R(\tau + \beta) = R(\tau)$$

$$R = \exp \left[ -\frac{i\phi}{2} \sigma_z \right] \exp \left[ -\frac{i\theta}{2} \sigma_y \right] \exp \left[ -\frac{i\psi}{2} \sigma_z \right] \quad \psi - \text{arbitrary}$$

$$\mathcal{S}_\Phi = \sum_\alpha \text{tr} \ln \left( -\partial_\tau - \epsilon_\alpha + \mu - \Phi(\tau) \frac{\sigma_z}{2} - R^{-1} \partial_\tau R \right)$$

# Adiabatic expansion, 0-th order

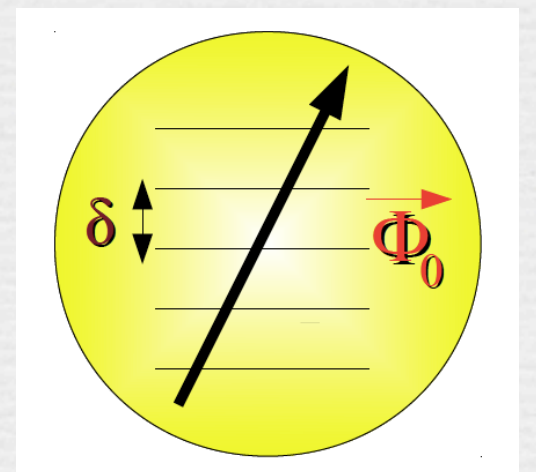
$\vec{\Phi}(\tau)$  large and slow

$$\mathcal{S}_{\Phi} = \sum_{\alpha} \text{tr} \ln \left( -\partial_{\tau} - \epsilon_{\alpha} + \mu - \Phi(\tau) \frac{\sigma_z}{2} - \cancel{R^{-1} \partial_{\tau} R} \right) - \int_0^{\beta} d\tau \frac{\Phi^2(\tau)}{4J}$$

$$-\beta\Omega(\Phi_0) = \sum_{\alpha} \text{tr} \ln \left( -\partial_{\tau} - \epsilon_{\alpha} + \mu - \Phi_0 \frac{\sigma_z}{2} \right) - \frac{\beta\Phi_0^2}{4J}$$

$$\Omega = \text{const.} + \left( \frac{1}{J} - \nu \right) \frac{\Phi_0^2}{4} = \text{const.} + \frac{\Phi_0^2}{4J_*}$$

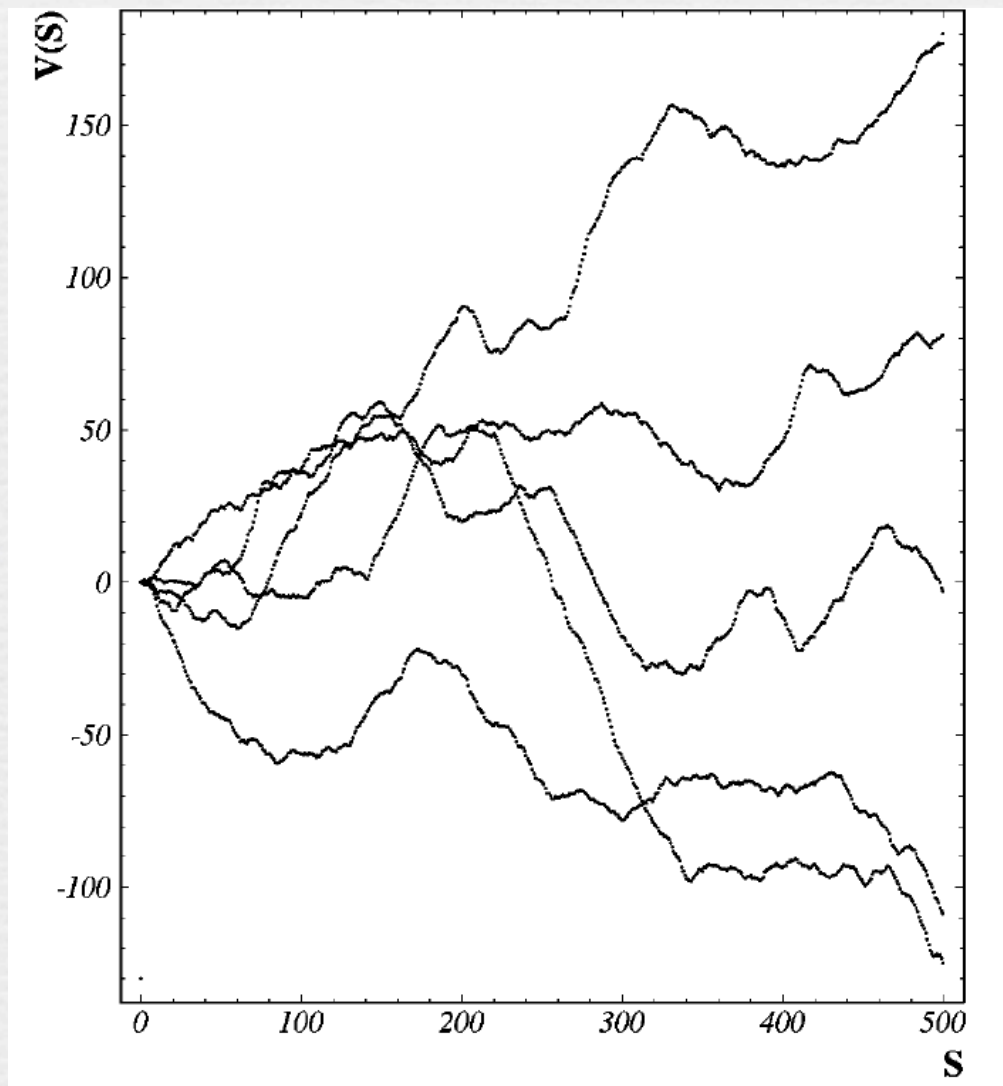
Stoner instability  $J_* = \frac{J}{1 - \nu J} \rightarrow \infty$



# Role of disorder

$$\frac{\partial \Omega}{\partial \Phi_0} = \frac{\Phi_0}{2} \left( \frac{1}{J} - \nu \right) - \frac{\delta n}{2} = \frac{\Phi_0}{2J_*} - \frac{\delta n}{2} \quad \bar{n} = \nu \Phi_0$$

$$\langle (\delta n)^2 \rangle \sim \ln \bar{n}$$



**L. Kurland, I. L. Aleiner, and B. L. Altshuler,**  
**Phys. Rev. B 62, 14886 (2000)**

$$\Omega = \frac{\Phi_0^2}{4J_*} + \delta\Omega$$

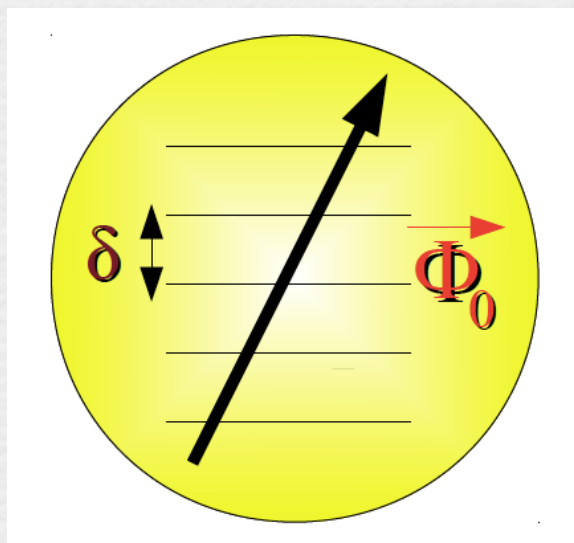
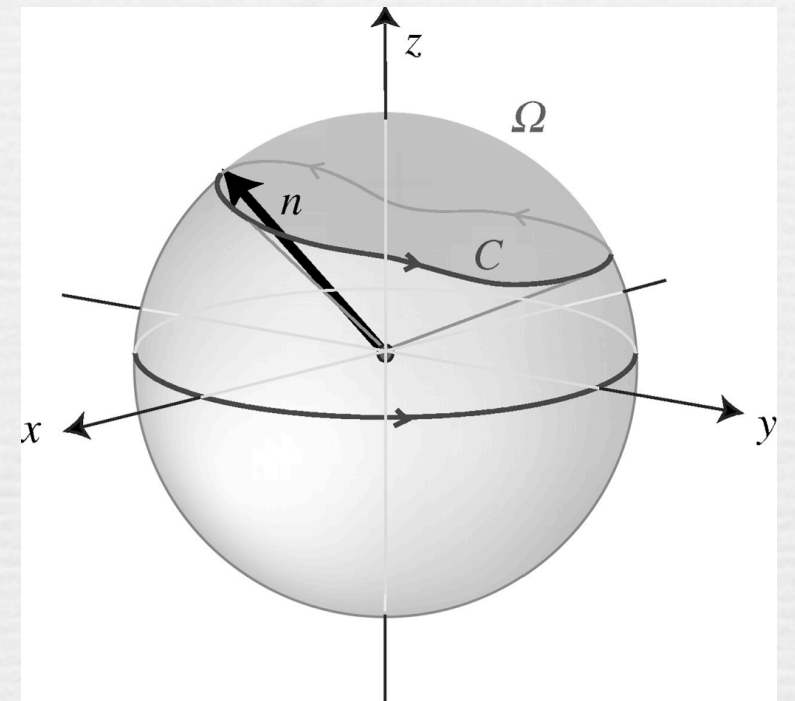
$$\delta\Omega \sim \pm O(1) |\Phi_0|$$

**I. S. Burmistrov, Y. Gefen, M. N. Kiselev,**  
**Phys. Rev. B 85, 155311 (2012)**

# Adiabatic, Expansion, 1-st order

$$\mathcal{S}_\Phi = \sum_\alpha \text{tr} \ln \left( -\partial_\tau - \epsilon_\alpha + \mu - \Phi(\tau) \frac{\sigma_z}{2} - R^{-1} \partial_\tau R \right)$$

$$\mathcal{S}_\Phi \approx -\beta \Omega(\Phi_0) + \frac{i\Gamma}{2} \int_0^\beta d\tau (1 - \cos \theta) \dot{\phi}$$



$$\Gamma = \nu \Phi_0 + \delta n$$

Disorder unimportant

Berry's phase  
WZNW action

# Integrating over Berry's phase

$$\mathcal{S}_\Phi \approx -\beta\Omega(\Phi_0) + \frac{i\Gamma}{2} \int_0^\beta d\tau (1 - \cos\theta) \dot{\phi}$$

$$\int \mathcal{D}\vec{n} \exp \left[ \frac{i\Gamma}{2} \int_0^\beta d\tau \dot{\phi} (1 - \cos\theta) \right]$$

Close to Stoner

$$\Gamma \gg 1$$

$$y \equiv 1 - \cos\theta \ll 1$$

$$\approx \int \mathcal{D}\phi \mathcal{D}y \exp \left[ \frac{i\Gamma}{2} \int_0^\beta d\tau \dot{\phi} y \right]$$

Mainly “small” contours  
contribute

$$\approx \prod_{m=1}^N \left( \frac{1}{\beta\Gamma\omega_m} \right)^2$$

Adiabatic condition

$$\omega_m \leq \Phi_0$$

Proper Jacobian crucial

# Integrating over Berry's phase

$$\mathcal{S}_\Phi \approx -\beta\Omega(\Phi_0) + \frac{i\Gamma}{2} \int_0^\beta d\tau (1 - \cos\theta) \dot{\phi}$$

$$\mathcal{Z} \propto \int_0^\infty d\Phi_0 4\pi\Phi_0^2 \exp\left[-\frac{\beta\Phi_0^2}{4J^*}\right] \cdot \frac{\sinh\left[\frac{\beta\Phi_0}{2}\right]}{\frac{\beta\Phi_0}{2}} \propto \left(\frac{J^*}{J}\right)^{3/2} \exp\left[\frac{\beta J^*}{4}\right]$$

$$\chi = \frac{1}{3} \frac{\partial \ln \mathcal{Z}}{\partial J} = \frac{1}{2} \frac{\nu}{(1 - J\nu)} + \frac{\beta}{12} \frac{1}{(1 - \nu J)^2}$$

Pauli

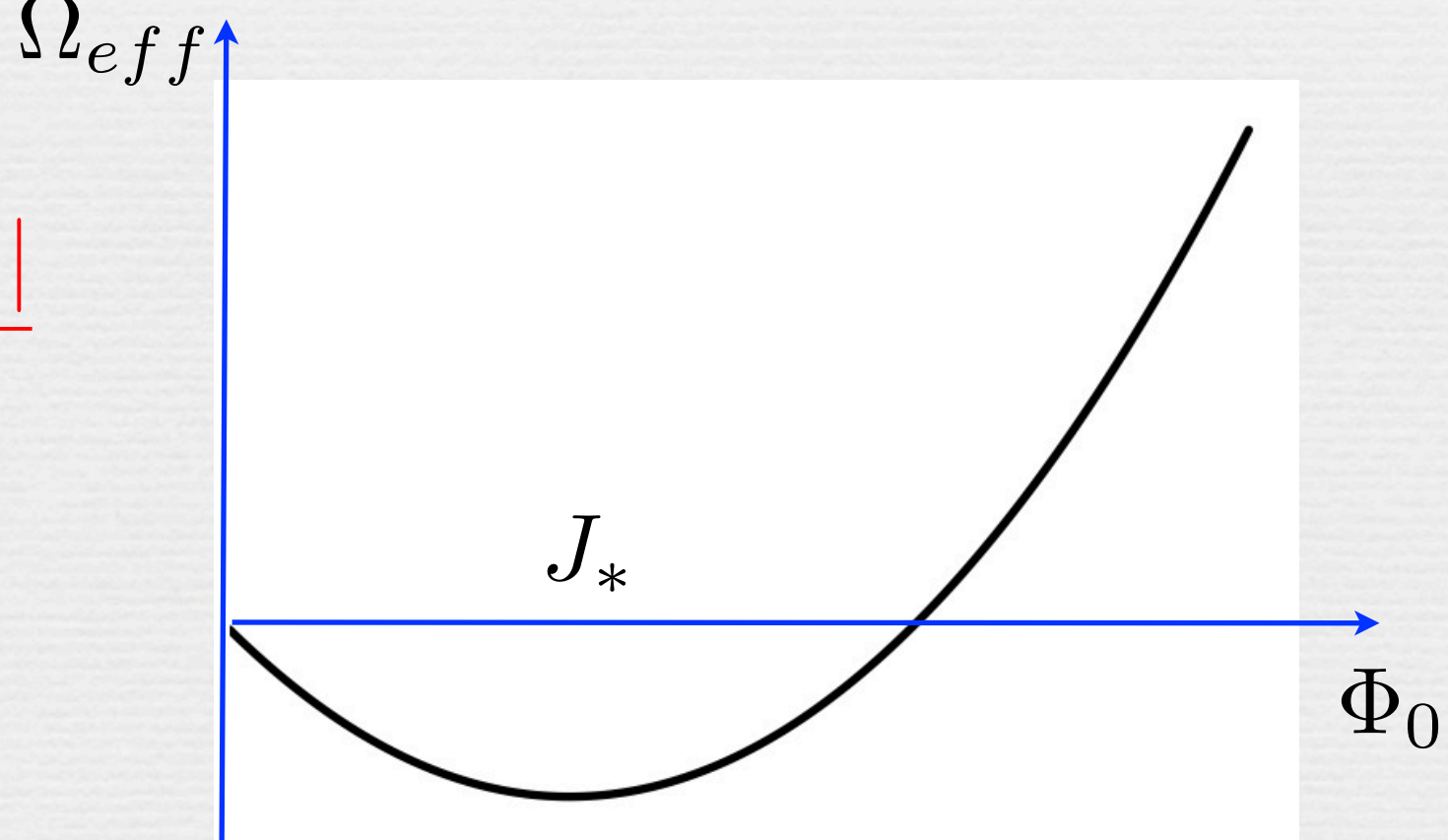
Curie

# Effective Potential

$$\mathcal{S}_\Phi \approx -\beta\Omega(\Phi_0) + \frac{i\Gamma}{2} \int_0^\beta d\tau (1 - \cos \theta) \dot{\phi}$$

$$\mathcal{Z} \propto \int_0^\infty d\Phi_0 4\pi \Phi_0^2 \exp \left[ -\frac{\beta \Phi_0^2}{4J^*} \right] \cdot \frac{\sinh \left[ \frac{\beta \Phi_0}{2} \right]}{\frac{\beta \Phi_0}{2}}$$

$$\Omega_{eff} \sim \frac{\Phi_0^2}{4J^*} - \frac{|\Phi_0|}{2}$$

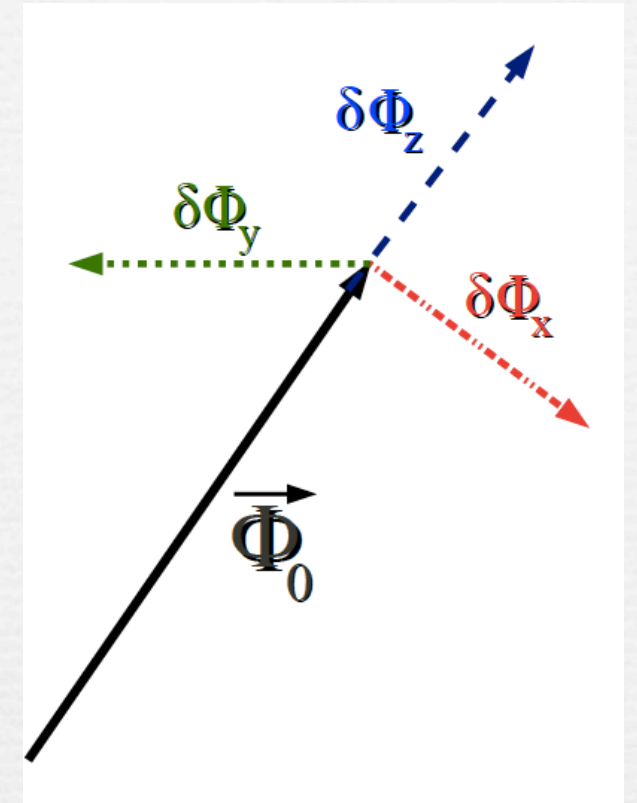


with disorder

$$\Omega_{eff} \sim \frac{\Phi_0^2}{4J^*} - \frac{|\Phi_0|}{2} \pm O(1)|\Phi_0|$$

# Cartesian coordinates

G.Murthy, Phys. Rev. B 77, 073309 (2008)

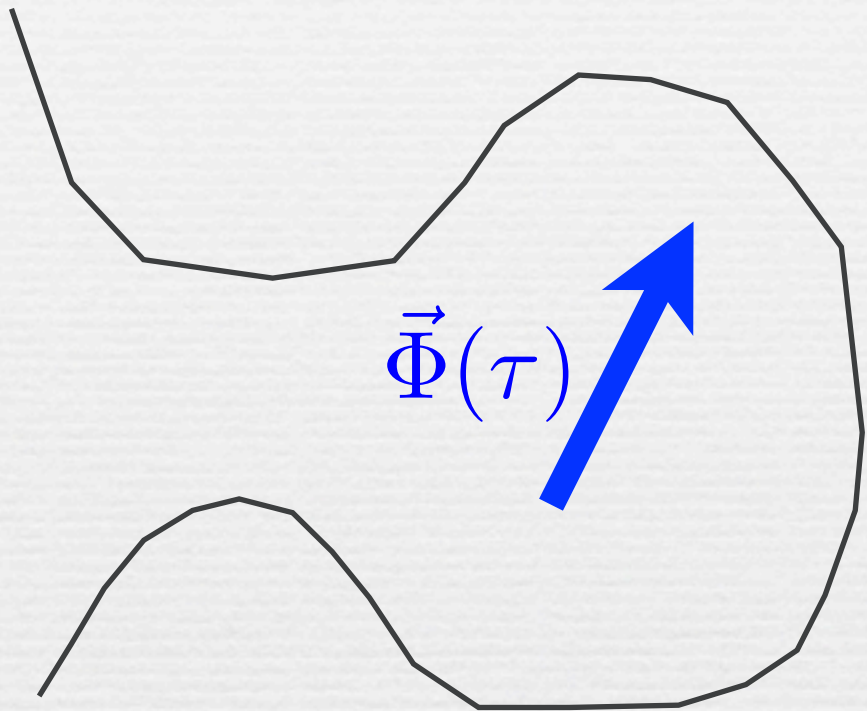


$$\mathcal{S}_{\Phi} \approx \text{const.} - \frac{\beta \Phi_0^2}{4J^*} - \sum_{m \neq 0} \frac{\beta \delta \vec{\Phi}_m \delta \vec{\Phi}_{-m}}{4J} - \sum_{\alpha} \frac{1}{2} \text{tr} \left[ G_0(\delta \vec{\Phi} \cdot \mathbf{S}) G_0(\delta \vec{\Phi} \cdot \mathbf{S}) \right]$$

Berry phase only for adiabatic paths

$$\mathcal{S}_{\delta \Phi} = \cdots - \sum_{m > 0} \frac{\beta \Gamma}{2} \frac{\omega_m}{\Phi_0^2 + \omega_m^2} (\delta \Phi_{x,m} \delta \Phi_{y,-m} - \delta \Phi_{y,m} \delta \Phi_{x,-m})$$

# Open dot, AES action



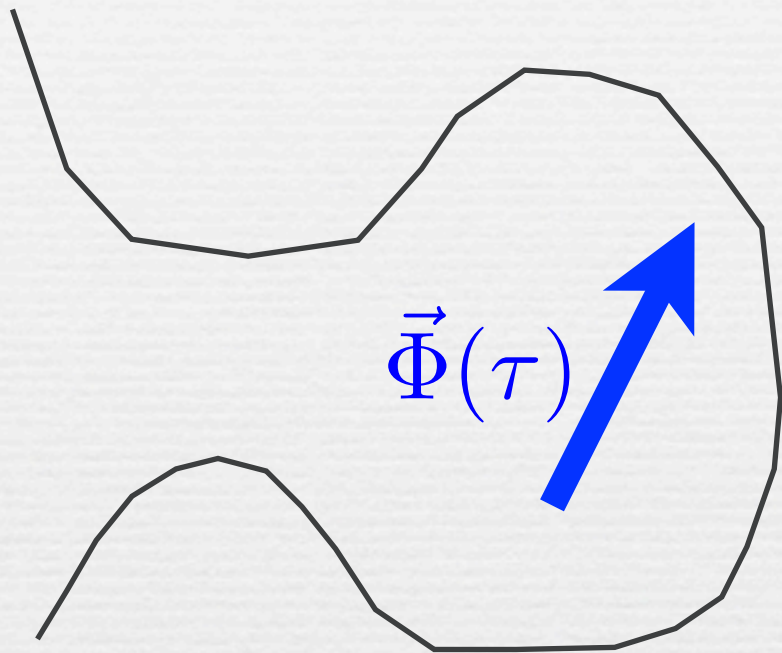
$$H = H_{dot} + H_{lead} + H_t$$

$$H_{dot} = \sum_{\alpha, \sigma} \epsilon_{\alpha} a_{\alpha, \sigma}^{\dagger} a_{\alpha, \sigma} - J \mathbf{S}^2$$

$$H_{lead} = \sum_{\gamma, \sigma} \epsilon_{\gamma} c_{\gamma, \sigma}^{\dagger} c_{\gamma, \sigma}$$

$$H_t = \sum_{\alpha, \gamma, \sigma} t_{\alpha, \gamma} a_{\alpha, \sigma}^{\dagger} c_{\gamma, \sigma} + h.c.$$

# Open dot, AES action



$$\mathcal{S}_{\Phi} = \text{tr} \ln \begin{pmatrix} G_{dot}^{-1} & t \\ t^{\dagger} & G_{lead}^{-1} \end{pmatrix} - \int_0^{\beta} d\tau \frac{|\vec{\Phi}|^2}{4J}$$

$$G_{dot}^{-1} = -\partial_{\tau} - \epsilon_{\alpha} - \vec{\Phi}(\tau) \cdot \vec{\mathbf{S}}$$

$$G_{lead}^{-1} = -\partial_{\tau} - \epsilon_{\gamma}$$

## Rotation

$$\mathcal{S}_{\Phi} = \text{tr} \ln \left[ \begin{pmatrix} R^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G_{dot}^{-1} & t \\ t^{\dagger} & G_{lead}^{-1} \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \right] - \int_0^{\beta} d\tau \frac{|\vec{\Phi}|^2}{4J}$$

$$= \text{tr} \ln \begin{pmatrix} R^{-1} G_{dot}^{-1} R & R^{-1} t \\ t^{\dagger} R & G_{lead}^{-1} \end{pmatrix} - \int_0^{\beta} d\tau \frac{|\vec{\Phi}|^2}{4J}$$

# Tunneling expansion

$$\mathcal{S}_\Phi = \text{tr} \ln \begin{pmatrix} R^{-1} G_{dot}^{-1} R & R^{-1} t \\ t^\dagger R & G_{lead}^{-1} \end{pmatrix} - \int_0^\beta d\tau \frac{|\vec{\Phi}|^2}{4J}$$

$$\begin{aligned} \text{tr} \ln (\dots) &= \text{tr} \ln (R^{-1} G_{dot}^{-1} R) + \text{tr} \ln (G_{lead}^{-1}) \\ &+ \sum_{k=1}^{\infty} \frac{1}{k} \text{tr} \left[ (R R^{-1} G_{dot} R R^{-1} t G_{lead} t^\dagger)^k \right] \end{aligned}$$

All orders separately gauge invariant

$$R \in SU(2)/U(1)$$

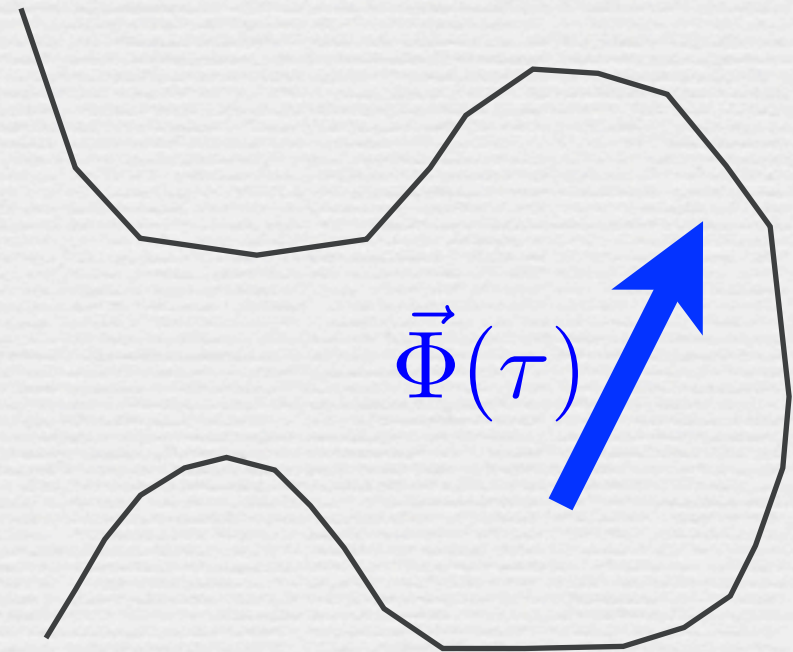
$$R = \exp \left[ -\frac{i\phi}{2} \sigma_z \right] \exp \left[ -\frac{i\theta}{2} \sigma_y \right] \exp \left[ -\frac{i\psi}{2} \sigma_z \right]$$

# Tunneling expansion

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{tr} \left[ \left( R R^{-1} G_{dot} R R^{-1} t G_{lead} t^{\dagger} \right)^k \right]$$

$$R^{-1} G_{dot} R = \left( R^{-1} G_{dot}^{-1} R \right)^{-1} = \left( G_{dot,z}^{-1} - R^{-1} \partial_{\tau} R \right)^{-1}$$

$$G_{dot,z}^{-1} = -\partial_{\tau} - \epsilon_{\alpha} - \Phi(\tau) S_z$$



# Problem with gauge invariance

$$R^{-1} G_{dot} R = \left( G_{dot,z}^{-1} - R^{-1} \partial_\tau R \right)^{-1} \approx G_{dot,z}$$

$$G_{dot,z}^{-1} = -\partial_\tau - \epsilon_\alpha - \Phi S_z$$

$$\mathcal{S}_t = \text{tr} \left( R G_{dot,z} R^{-1} t G_{lead} t^\dagger \right)$$

$$\mathcal{S}_t = \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \Lambda(\tau_1 - \tau_2) \text{tr} \left[ R(\tau_1) R^{-1}(\tau_2) \right]$$

$$\Lambda(\tau) = -\frac{\pi^2 \rho^{lead} \rho^{dot} |t|^2}{\sin^2(\pi T \tau)}$$

# Problem with gauge invariance

$$\mathcal{S}_t = \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \Lambda(\tau_1 - \tau_2) \text{tr} [R(\tau_1) R^{-1}(\tau_2)]$$

$$\Lambda(\tau) = -\frac{\pi^2 \rho^{lead} \rho^{dot} |t|^2}{\sin^2(\pi T \tau)}$$

$$R = \exp \left[ -\frac{i\phi}{2} \sigma_z \right] \exp \left[ -\frac{i\theta}{2} \sigma_y \right] \exp \left[ -\frac{i\psi}{2} \sigma_z \right]$$

$$\text{tr} [R(\tau_1) R^{-1}(\tau_2)] =$$

$$\begin{aligned} & \cos \frac{\theta(\tau_1)}{2} \cos \frac{\theta(\tau_2)}{2} \cos \left( \frac{\phi(\tau_1) - \phi(\tau_2)}{2} + \frac{\psi(\tau_1) - \psi(\tau_2)}{2} \right) \\ & + \sin \frac{\theta(\tau_1)}{2} \sin \frac{\theta(\tau_2)}{2} \cos \left( \frac{\phi(\tau_1) - \phi(\tau_2)}{2} - \frac{\psi(\tau_1) - \psi(\tau_2)}{2} \right) \end{aligned}$$

# Gauge invariant approach

$$\mathcal{S}_t = \sum_{k=1}^{\infty} \frac{1}{k} \text{tr} \left[ \left( R \left[ G_{\dot{\theta}, z}^{-1} - R^{-1} \partial_{\tau} R \right]^{-1} R^{-1} t G_{lead} t^{\dagger} \right)^k \right]$$

$$R^{-1} \partial_{\tau} R = -\frac{i}{2} (\dot{\phi} \cos \theta + \dot{\psi}) \sigma_z$$

~~$$-\frac{i}{2} \exp \left[ \frac{i\psi}{2} \sigma_z \right] \left[ \dot{\theta} \sigma_y - \dot{\phi} \sin \theta \sigma_x \right] \exp \left[ -\frac{i\psi}{2} \sigma_z \right]$$~~

$$R^{-1} \partial_{\tau} R \approx \frac{i}{2} \chi(\tau) \sigma_z$$

$$\int_0^{\beta} d\tau \chi(\tau) = \Phi_B$$

Berry phase,  
zero mode

$$\chi(\tau) = \frac{\Phi_B}{\beta} + \delta\chi(\tau)$$

Gauge out !

# Gauge invariant approach

$$\mathcal{S}_t = \sum_{k=1}^{\infty} \frac{1}{k} \text{tr} \left[ \left( R \left[ G_{dot,z}^{-1} - \frac{i\Phi_B}{\beta} \sigma_z \right]^{-1} R^{-1} t G_{lead} t^\dagger \right)^k \right]$$

$$R^{-1} \partial_\tau R \approx -\frac{i}{2} (\dot{\phi} \cos \theta + \dot{\psi}) \sigma_z = \frac{i}{2} \frac{\Phi_B}{\beta} \sigma_z$$

$$\dot{\phi} \cos \theta + \dot{\psi} = -\frac{\Phi_B}{\beta} \quad \text{gauge fixed} \quad -2\pi < \Phi_B < 2\pi$$

$$\mathcal{S}_t^{k=1} = \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \Lambda(\tau_1 - \tau_2) \text{tr} [R(\tau_1) R^{-1}(\tau_2)]$$

$$\Lambda(\tau) = -\frac{\pi^2 \rho^{lead} \rho^{dot} |t|^2}{\sin^2(\pi T \tau)}$$

# Gauge invariant AES action

$$\mathcal{S}_t^{k=1} = \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \Lambda(\tau_1 - \tau_2) \text{tr} [R(\tau_1) R^{-1}(\tau_2)]$$

$$\Lambda(\tau) = -\frac{\pi^2 \rho^{lead} \rho^{dot} |t|^2}{\sin^2(\pi T \tau)}$$

$$\text{tr} [R(\tau_1) R^{-1}(\tau_2)] =$$

$$\cos \frac{\theta(\tau_1)}{2} \cos \frac{\theta(\tau_2)}{2} \cos \left( \frac{\phi(\tau_1) - \phi(\tau_2)}{2} + \frac{\psi(\tau_1) - \psi(\tau_2)}{2} \right)$$

$$+ \sin \frac{\theta(\tau_1)}{2} \sin \frac{\theta(\tau_2)}{2} \cos \left( \frac{\phi(\tau_1) - \phi(\tau_2)}{2} - \frac{\psi(\tau_1) - \psi(\tau_2)}{2} \right)$$

**gauge fixed**

$$\dot{\phi} \cos \theta + \dot{\psi} = -\frac{\Phi_B}{\beta}$$

$$\Phi_B \equiv \int_0^\beta d\tau \dot{\phi} (1 - \cos \theta) + 4\pi N$$

$$-2\pi < \Phi_B < 2\pi$$

In real time (Keldysh)  
Gilbert friction!

some extra term

# Conclusions

- Adiabatic Berry phase action reproduces the partition function (susceptibility) of a closed dot close to Stoner instability
- Open dots, AES action, gauge invariance important, Gilbert friction