Probability of high intensities of the light wave propagating in turbulent media

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The atmosphere is always turbulent due to the high value of the Reynolds number

$$\mathrm{Re} = LV/\mu.$$

Here L is the scale, where turbulence is excited, V is the characteristic velocity at the scale, and μ is kinematic viscosity, $1.5 \cdot 10^{-5}\, m/s^2$ for the normal conditions.

Turbulent pulsations cause fluctuations of the refractive index ν. The fluctuation ν is a random variable. That means that ν chaotically varies in time and space. Therefore its properties has to be described statistically, in terms of averages over time or/and space. To characterize the statistical properties well, one has to average over a big massive of data. Different snapshots $\nu(t, \boldsymbol{r})$ can differ essentially.

We designate the averages by angular brackets. Say, the average value of ν^2 is designated as $\langle \nu^2 \rangle$. We are counting down ν from its average, therefore $\langle \nu \rangle = 0$. Statistical properties of ν can be expressed in terms of its momenta $\langle \nu^n \rangle$. The first moment is equal to zero. The second moment is positive as well all even moments. Odd moments can be either positive and negative.

One introduces an alternative way to characterize statistical properties of a random variable x. One introduces its probability density function $P(x)$. By definition, $P(x)\,dx$ is the probability to find the random variable in the interval between x and $x + dx$. Of course

$$\int dx\, P(x) = 1,$$

that is the total probability is equal to unity.

To find the probability density function $P(x)$ one can find first the histogram of the variable x. For this the region of existing x is divided into intervals of length Δx and one counts a number of x falling into the interval. In the limit $\Delta x \to 0$ one finds a smooth curve. Normalizing the curve one finds the probability density function $P(x)$. Big data are needed!

If the probability density function $P(x)$ of x is known then one can find its moments

$$\langle x^n \rangle = \int dx\, P(x) x^n .$$

Normal (Gaussian) probability density

$$P(x) = (2\pi)^{-1/2} b^{-1} \exp\left[-x^2/(2b^2)\right],$$
$$\langle x^{2n} \rangle = (2n)!(2^n n!)^{-1} b^{2n}.$$

Odd moments are zero.

Why Gaussian probability density is so important?

Central limit theorem: if x is a sum of a big number of statistically independent random variables then it possesses Gaussian probability density irrespective to statistical properties of the variables.

Example: Maxwell probability density of the molecular velocities of an ideal gas. Reason: we consider the average velocity determined by the sum over a big number of molecules.

Fat tails of the density function $P(x)$: probability of strong deviations are much larger than for the normal PDF. Usually such systems are characterized by tails with stretched exponents

$$\ln P \propto -x^{\beta},$$

for large x. The less is the index β the more fat the tail is. The case $\beta \to 0$ corresponds to power tails.

The high moments x are determined by fat tails. For the tail characterized by the stretched exponent

$$\langle x^n \rangle \propto n^{n/\beta}.$$

Again, the less is the index β the more are the high moments normalized by the second moment. Extreme case: for power PDF the moments are infinite starting from some number.

Turbulence is a chaotic state with a random velocity $v(t, r)$, which should be characterized statistically, that is, through average values, which we denote with angular brackets. The averages can be calculated over time and/or over the space region where turbulence is excited. For a field, one should consider correlation functions, say $\langle v_1 v_2 \rangle$, instead of the moments.

On scales much smaller than L, turbulence is homogeneous and isotropic. Traditionally, the properties of turbulence are characterized by the simultaneous pair correlation function of the velocity

$$\langle \boldsymbol{v}(t, \boldsymbol{R}_1) \boldsymbol{v}(t, \boldsymbol{R}_2) \rangle.$$

It depends solely on the absolute value R of the separation $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$ on scales much smaller than L, $R \ll L$.

The correlation function can be represented as a Fourier integral

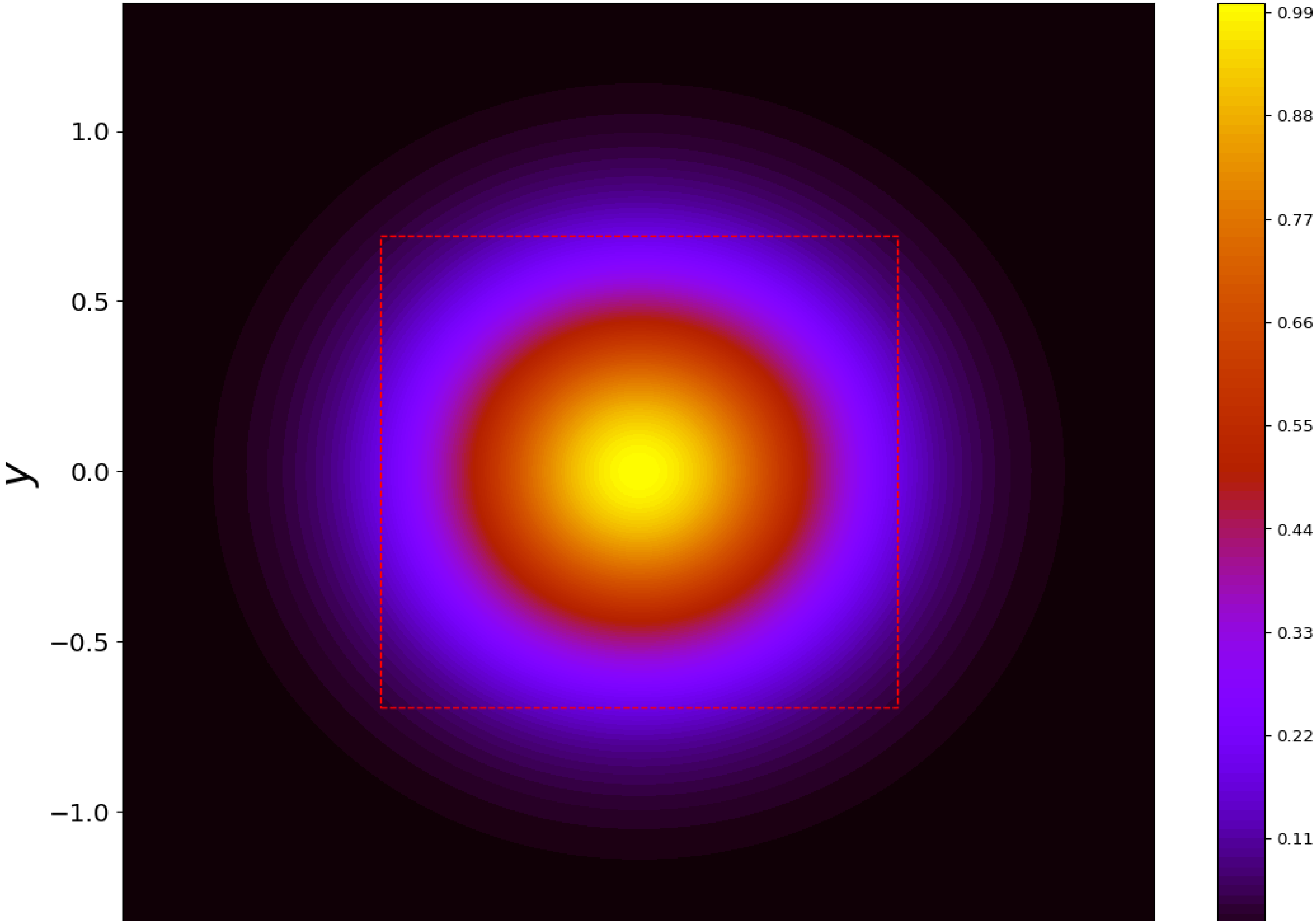
$$\int \frac{dk}{2\pi} \exp(ikR) E(k),$$

where $E(k)$ is called the turbulence spectrum. The spectrum possesses a scaling behavior. In the framework of Kolmogorov theory $E(k) \propto k^{-5/3}$. The Kolmogorov spectrum fits the observations pretty well.

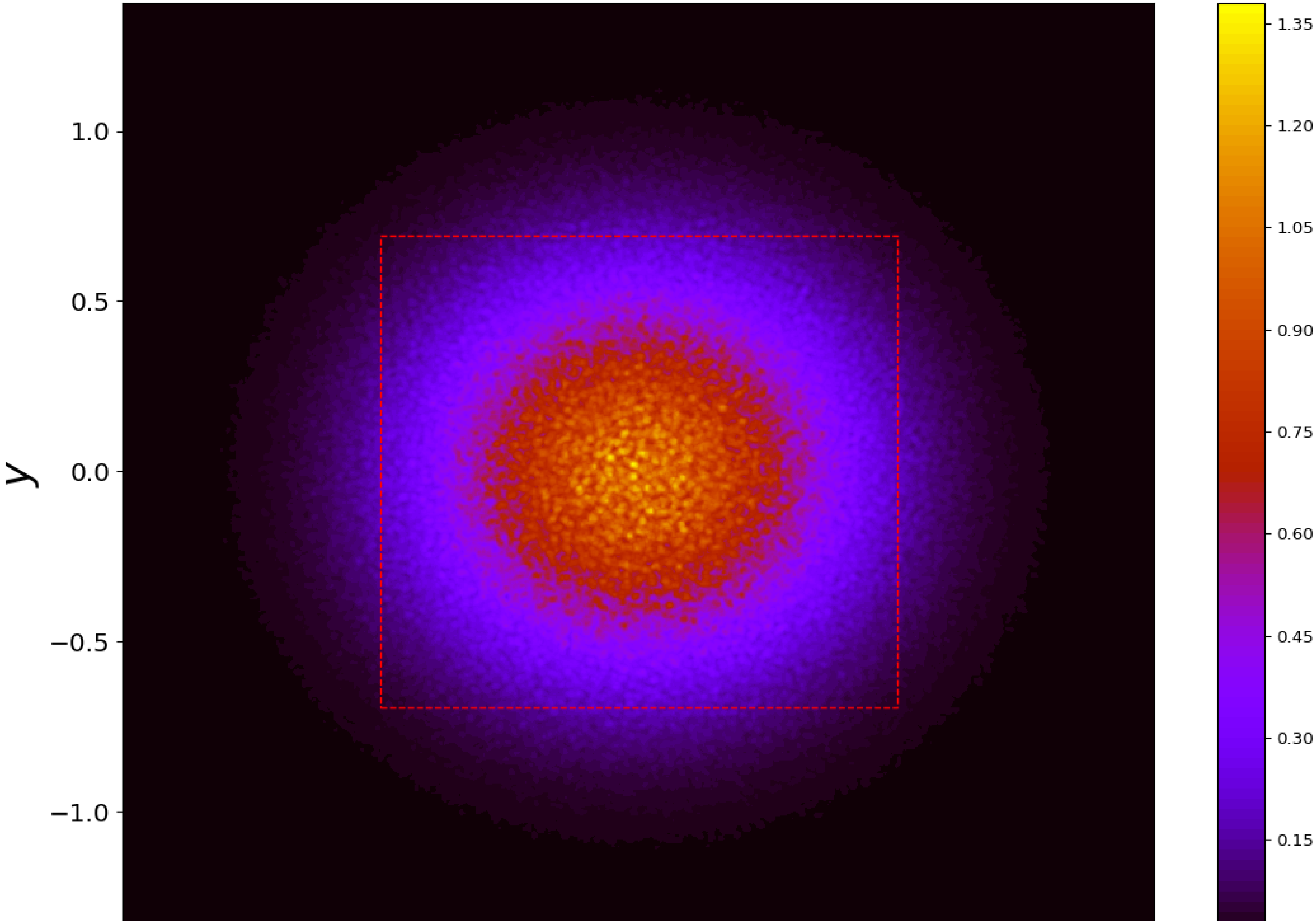
The physical mechanism behind the power spectrum is the energy cascade. Fluid motions exited by an external pumping at the scale L generate movements on a smaller and smaller scale until this process stops due to viscosity. This is accompanied by the transfer of the kinetic energy from the scale L to small scales, where it passes to heat due to viscosity.

There are passive fields (temperature, concentration of impurities) advected by the turbulent velocity. As a consequence, they becomes chaotic and multiscaled. They should be characterized by their own spectrum. In accordance with Obukhov-Corrsin theory the spectrum is characterized by the same exponent $5/3$ as the velocity spectrum. However, the range boundary is determined by diffusion.

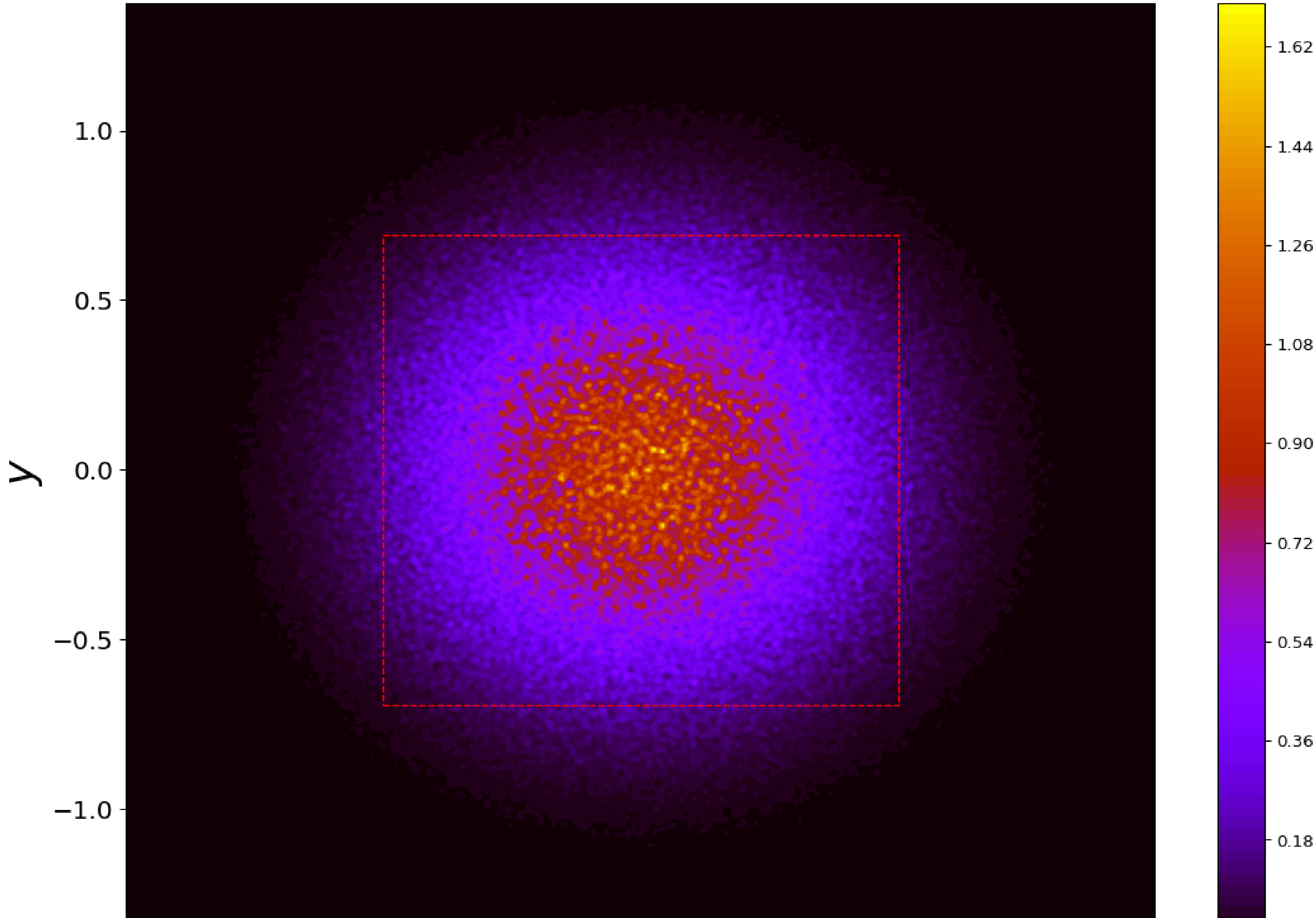
A light beam propagating in the atmosphere diffracts on fluctuations in the refractive index and is distorted, as a result. Gradually, it falls apart into speckles. Let's illustrate the character of these distortions using the example of a beam that originally had a Gaussian profile. We give intensity profiles plotted in the transverse plane at a distance from the source.



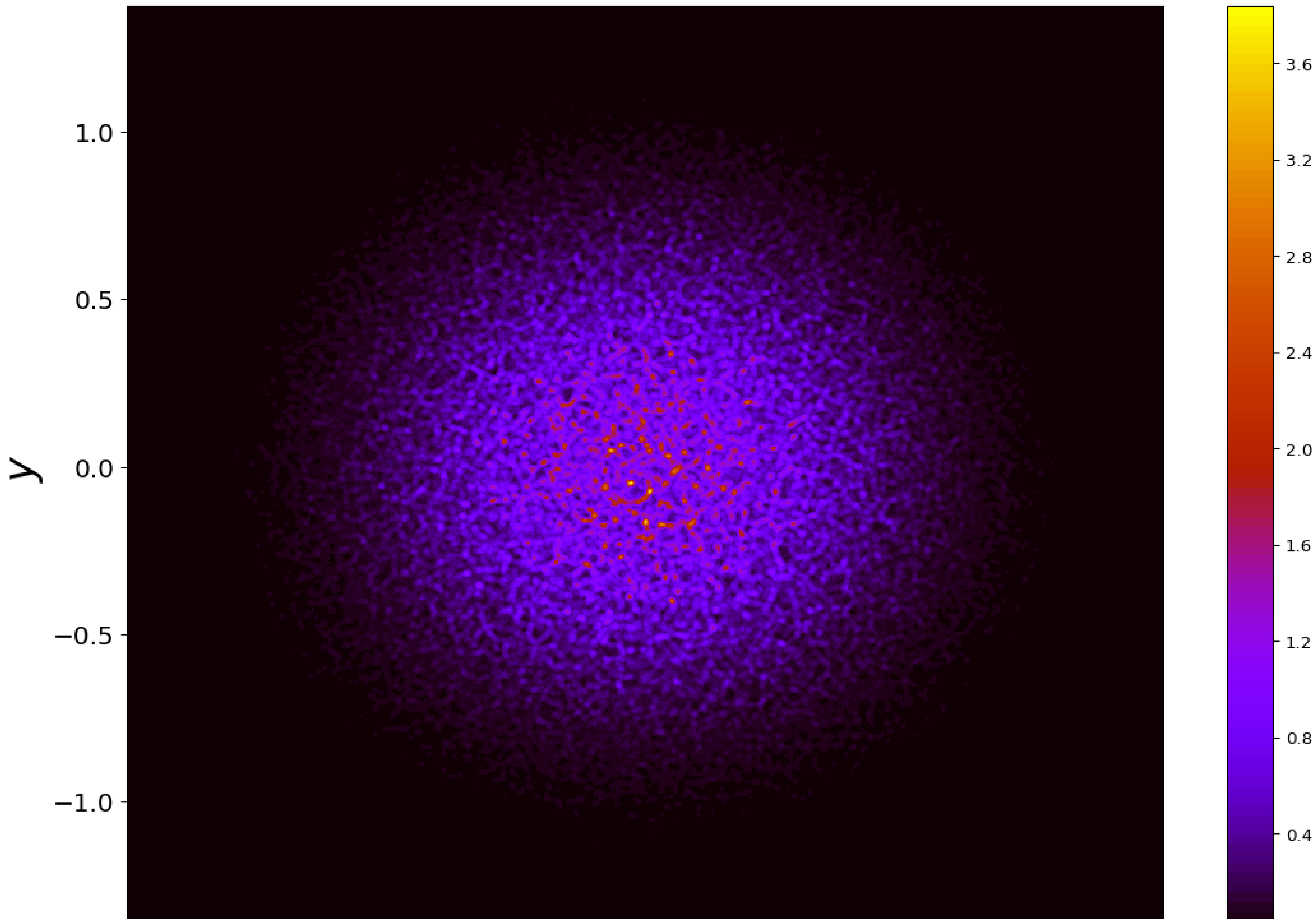
1.0
0.5
0.0
−0.5
−1.0
y
0.99
0.88
0.77
0.66
0.55
0.44
0.33
0.22
0.11



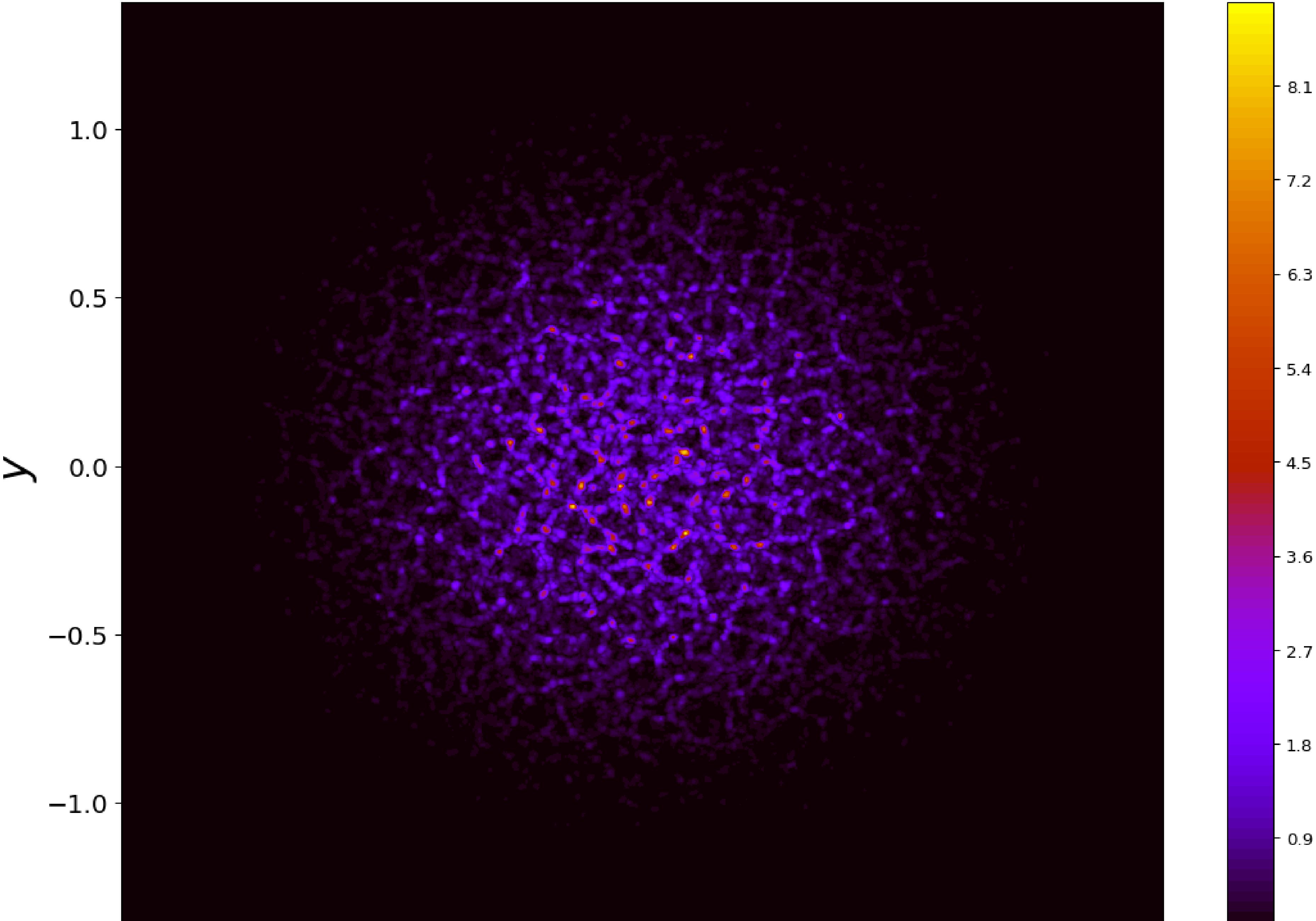
1.0
0.5
0.0
−0.5
−1.0
y
1.35
1.20
1.05
0.90
0.75
0.60
0.45
0.30
0.15



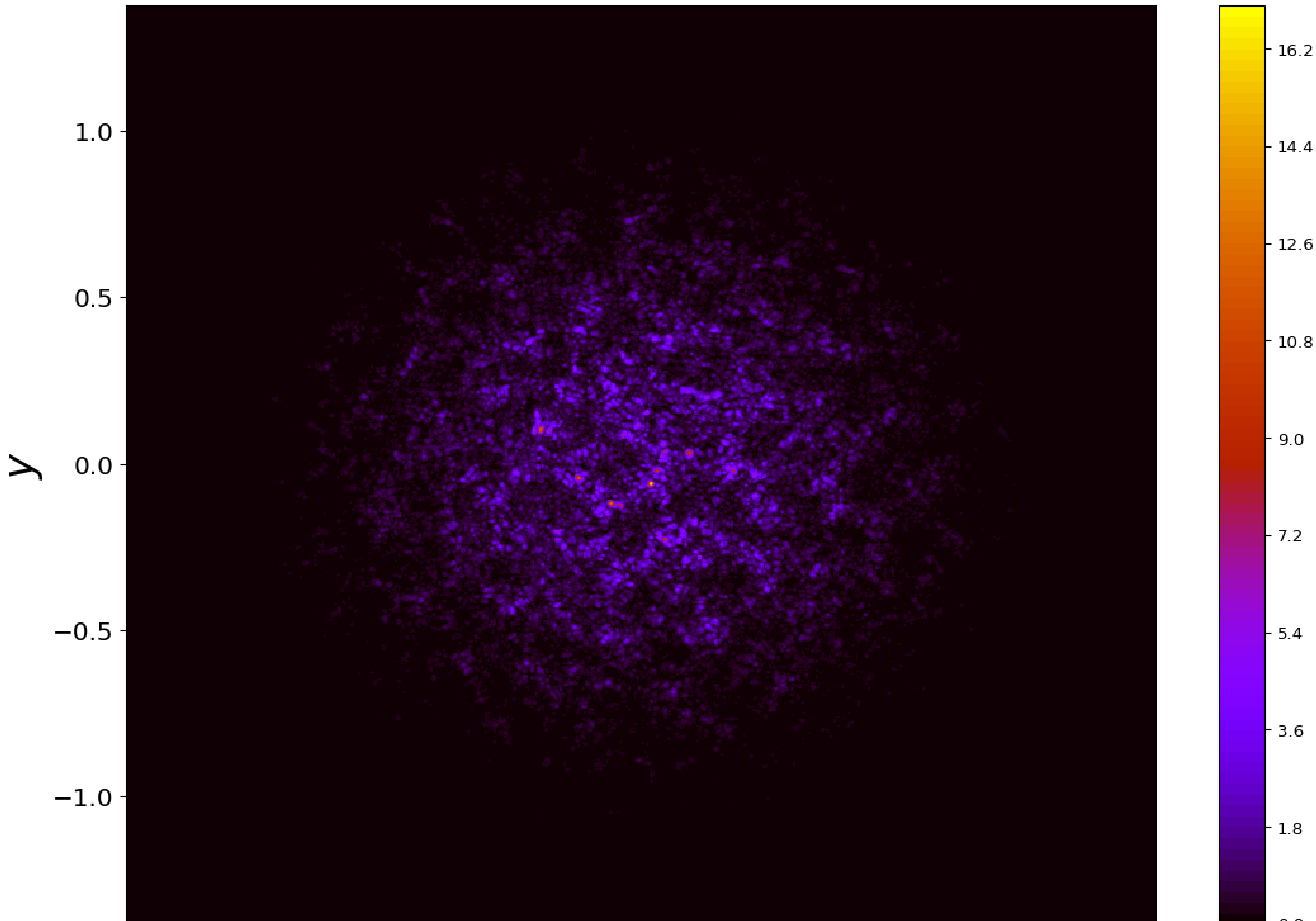
1.0
0.5
0.0
−0.5
−1.0
y
1.62
1.44
1.26
1.08
0.90
0.72
0.54
0.36
0.18



y
1.0
0.5
0.0
−0.5
−1.0
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4



1.0
0.5
0.0
−0.5
−1.0
y
8.1
7.2
6.3
5.4
4.5
3.6
2.7
1.8
0.9



y
1.0
0.5
0.0
−0.5
−1.0
16.2
14.4
12.6
10.8
9.0
7.2
5.4
3.6
1.8

For a theoretical description of the propagation of a light wave, the equation for the envelope Ψ should be used, which determines the electric field of the wave equal to

$$\mathrm{Re}\,[\Psi\,\exp(ik_0 z - i\omega_0 t)]\,.$$

Here k_0 is the wave vector, ω_0 is the frequency of the carrying wave and the wave propagates along Z-axis.

The envelope Ψ depends on time t and coordinates $\boldsymbol{r}, z$. Due to the high speed of light, the state of the atmosphere is unchanged during the passage of the beam. For example, to pass 3 km, it takes 10^{-5}s. Therefore, Ψ adiabatically adjusts to the current state of the atmosphere, which is determined by turbulent motions.

The envelope equation can be obtained from Maxwell's equations, it has the form

$$i\partial_z \Psi + \frac{1}{2k_0} \nabla^2 \Psi + k_0 \nu \Psi = 0.$$

Here ν is the fluctuation of the refractive index and ∇ is the derivative in the transverse direction. Due to the adiabaticity, there is no time derivative in this equation.

The fluctuation ν is a random variable that should be described statistically, in terms of averages. We are counting down ν from the average value of the refractive index, therefore $\langle\nu\rangle = 0$. We are interested in the evolution of Ψ along the Z axis when light travels a long distance quickly. Therefore, ν changes rapidly along the Z-axis.

The Obukhov-Corrsin spectrum lead to the conclusion that the structure function

$$\langle [\nu(\boldsymbol{r}, z+\zeta) - \nu(\mathbf{0}, z)]^2 \rangle = C_n^2 (r^2 + \zeta^2)^{1/3},$$

where the coefficient C_n characterizes the strength of the refractive index fluctuations. Here $\boldsymbol{r}$ is the two-dimensional radius-vector in the transverse direction. Generally, C_n is a function of z.

Since the field ν is effectively short correlated along Z one can substitute

$$\langle \nu(\boldsymbol{r}, z)\nu(\mathbf{0}, 0)\rangle \to \delta(z) C_n^2 (\mathcal{C} - 1.4572\, r^{5/3}),$$

where the constant $\mathcal{C}$ is determined by small q (of the order of the inverse external scale of turbulence). The same short correlation means that the field ν possesses Gaussian statistics due to CLT: enters via integrals.

Numerical simulations: a number of plane screens perpendicular to the Z-axis are introduced. Between the screens, the envelope Ψ evolves free. On screens, the phase of the envelope gains a random increment. It is a sum of Fourier harmonics guarantein

$$\langle \Delta\varphi(\boldsymbol{r})\Delta\varphi(\boldsymbol{0})\rangle = C_n^2(\mathcal{C} - 1.4572\, r^{5/3})\Delta z,$$

where Δz is the separation between the screens.

Due to the presence of ν, the envelope of Ψ is distorted compared to the case of free propagation. If these distortions are small, then the correction $\delta\Psi$ to the solution of the free equation can be written as a spatial integral of ν. Therefore, by virtue of the central limit theorem, the correction $\delta\Psi$ has Gaussian statistics, provided the inequality $|\delta\Psi| \ll |\Psi|$ is satisfied.

The equation for the envelope Ψ and the short correlation of ν along the axis Z allow us to obtain a closed differential equation for a pair correlation function

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2, z) = \langle \Psi(\boldsymbol{r}_1, z) \Psi^\star(\boldsymbol{r}_2, z) \rangle,$$

where $\Psi^\star$ is the quantity complex conjugated to Ψ. The quantity is insensitive to large scale fluctuations of ν.

The equation is as follows

$$\partial_z F = \frac{i}{2k_0}(\nabla_1^2 - \nabla_2^2)F - D|\boldsymbol{r}_1 - \boldsymbol{r}_2|^{5/3}F,$$

where the factor D depends on the power of turbulent pulsations $D \sim k_0^2 C_n^2$. Generally, D is a function of z. Note the non-locality of the equation. Nevertheless, it allows for detailed research.

For example, for an initial plane wave

$$F \propto \exp\left[-\int^{z} d\zeta\, D(\zeta)|\boldsymbol{r}_1 - \boldsymbol{r}_2|^{5/3}\right].$$

One can introduce the envelope correlation length r_0 (Fried length) equating the argument of the exponent to unity. As the beam propagates, r_0 decreases. This is a consequence of the increasing diffraction effect on ν.

Similarly, it is possible to obtain closed equations for higher correlation functions of the envelope. For example, the equation for the four-point correlation function

$$\langle \Psi(r_1, z)\Psi(r_2, z)\Psi^\star(r_3, z)\Psi^\star(r_4, z)\rangle.$$

Unfortunately, the study of the high-order equations encounters significant technical difficulties.

Further, we focus on the study of the statistical properties of the intensity of the electromagnetic wave $I = |\Psi|^2$. Unfortunately, it is impossible to find a closed equation on I. Therefore, we must inevitably first investigate correlations of the envelope Ψ, and then use the results to analyze statistics of I.

It is convenient to exploit the following dimensionless parameter

$$\gamma = z/(k_0 r_0^2) \sim D^{6/5} z^{11/5} / k_0,$$

absorbing both, the random diffraction and the geometrical characteristics. The case $\gamma \ll 1$ correspon to the regime of weak scintillations and the regime of strong scintillations is realized if $\gamma \gg 1$.

In the regime of weak scintillations, the wave is slightly distorted due to random diffraction. In this case, I is close to the value for free wave propagation. The correction caused by diffraction

$$\delta I = \Psi^{\star}\delta\Psi + \delta\Psi^{\star}\Psi,$$

possesses Gaussian statistics if $|\delta I| \ll I$ since both, $\delta\Psi$ and $\delta\Psi^{\star}$, possess this statistics.

In the regime of strong scintillations, Ψ is formed due to interference of signals from many random diffraction centers. By virtue of the central limit theorem, Ψ acquires Gaussian statistics, that is

$$P(\Psi) \propto \exp\left(-|\Psi|^2/I_0\right).$$

Here P is the probability density function (PDF) of Ψ.

In the argument of this exponent, there is nothing else than $I = |\Psi|^2$. Therefore, for intensity, the probability density has the form

$$P(I) = I_0^{-1} \exp\left(-I/I_0\right),$$

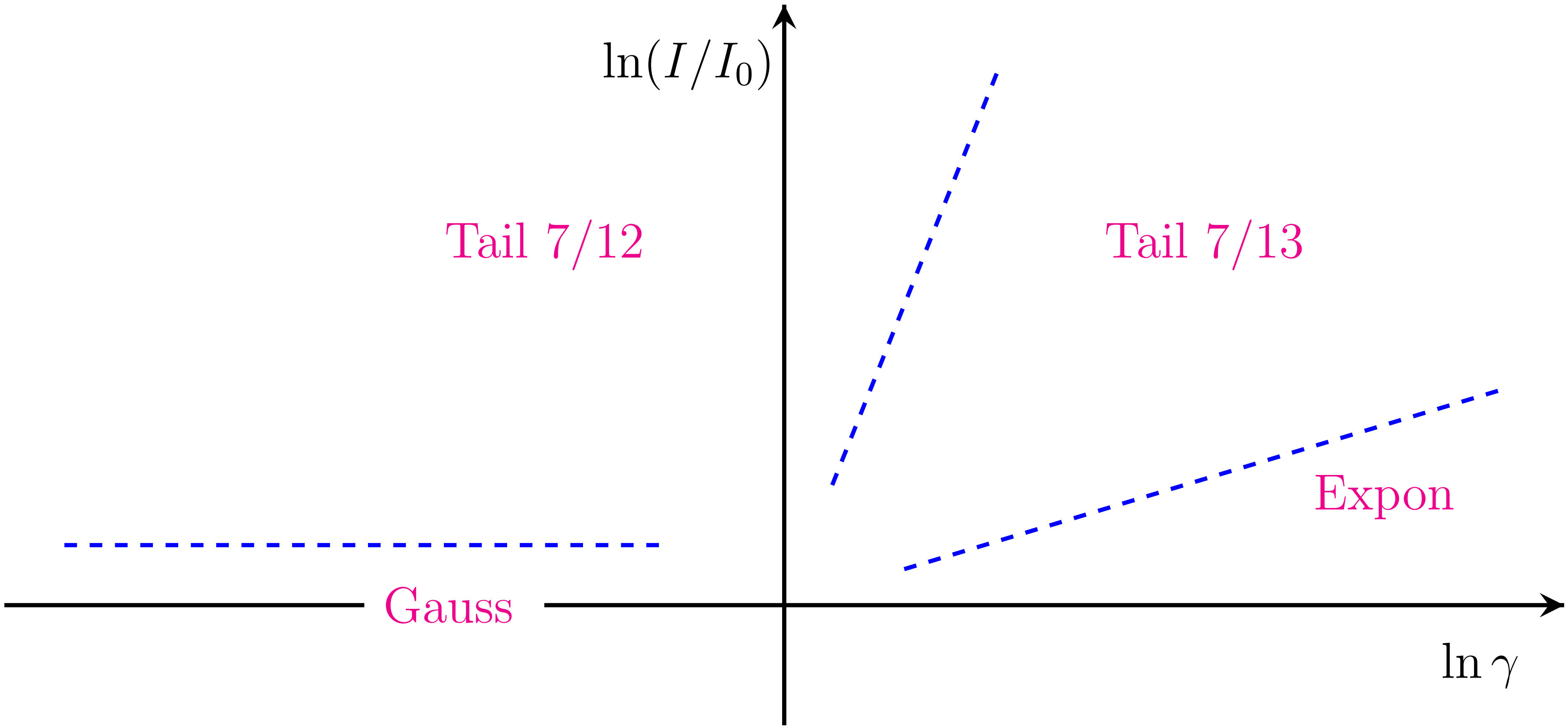
where I_0 is the average intensity at the observation point. However, it covers a restricted range of I.

The random variable ν enters the equation

$$i\partial_z \Psi + \frac{1}{2k_0}\nabla^2 \Psi + k_0 \nu \Psi = 0,$$

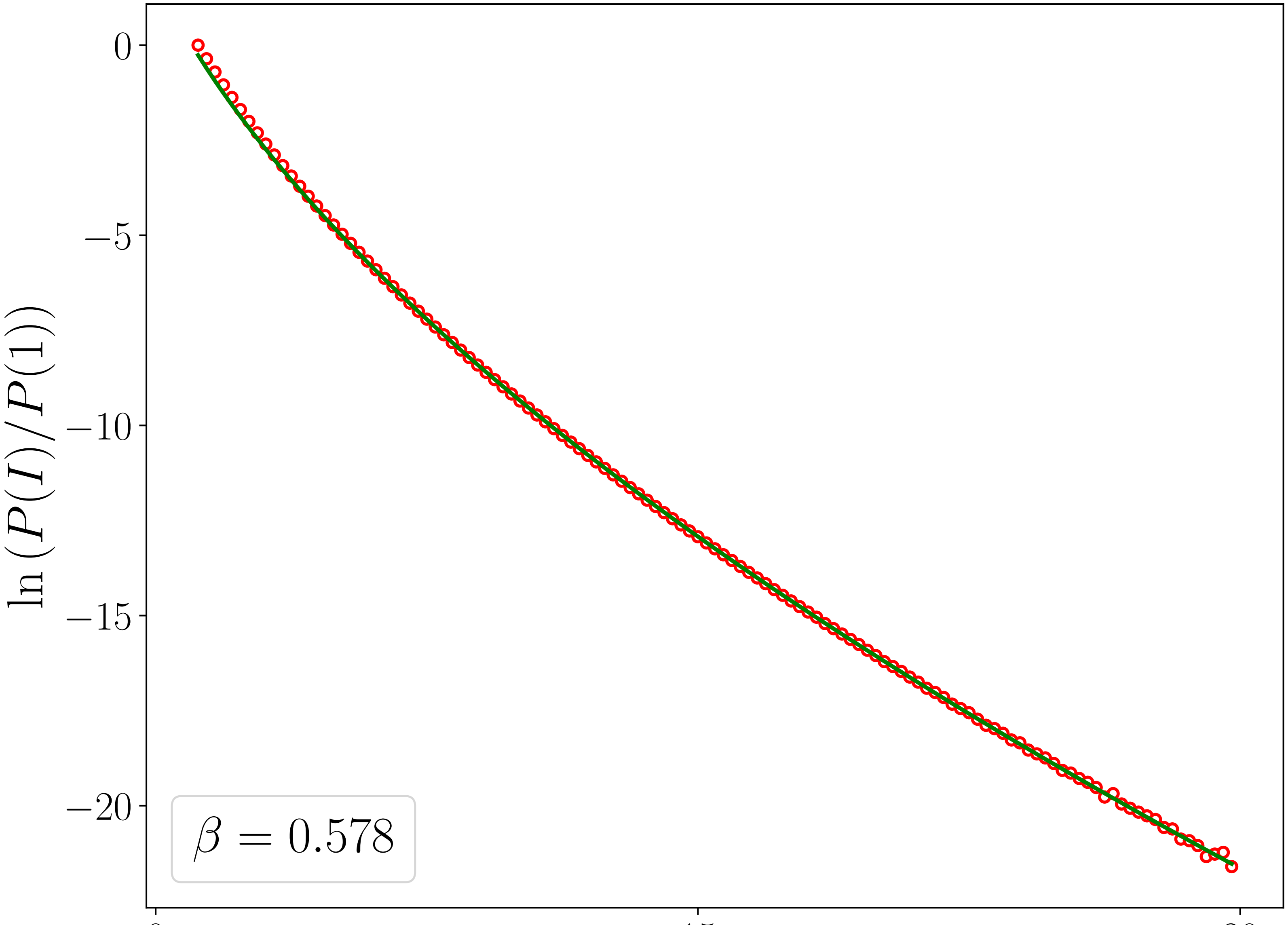
as a factor at Ψ. In this case, one expects an abnormally high probability of large values of $|\Psi|$ or of $I = |\Psi|^2$. The probability is manifested in "fat" tails of PDF, characterized by the stretched exponents: $\ln P(I) \propto -I^{\beta}$.

There are even two such tails. The first of them has to be observed well in the regime of moderate scintillations, $\beta = 7/12$. The second of them is implemented in the regime of strong scintillations, $\beta = 7/13$. Both exponents are less than one, that is the tails are “fat”, indeed. The overall picture is reflected in the diagram below.



$\ln(I/I_0)$
Tail 7/12
Tail 7/13
Expon
Gauss
$\ln\gamma$

We performed numerical simulations of $P(I)$. We used the scheme with single screen and with a number of screens. Each screen has a random refraction index with statistics determined by the Kolmogorov spectrum. The problem is to gain enough statistics. For the purpose a lot of realizations is used. We present results for the case of moderate scintillations, $\gamma \sim 1$.



$\ln(P(I)/P(1))$
0
−5
−10
−15
−20
$\beta = 0.578$

Here a dependence of $\ln P$ on I/I_0 is plotted. Red circles – average values, green line corresponds to the best fit to a stretched exponent giving the value $\beta = 0.578$, close to $\beta = 7/12 = 0.583$. We see that the tail starts immediately from $I/I_0 \sim 1$. To extract the tail for the cases of weak and strong scintillations, much more statistics is needed.