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FINITE-TEMPERATURE PERTURBATION THEORY
FOR THE RANDOM DIRECTED POLYMER PROBLEM

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Dedicated to the memory of Professor Anatoly Larkin

We study the random directed polymer problem — the short-scale behavior of an elastic string (or polymer) in one transverse dimension subject to a disorder potential and finite temperature fluctuations. We are interested in the polymer short-scale wandering expressed through the displacement correlator $\langle [\delta u(X)]^2 \rangle$, with $\delta u(X)$ being the difference in the displacements at two points separated by a distance X. While this object can be calculated at short scales using the perturbation theory in higher dimensions $d > 2$, this approach becomes ill-defined and the problem turns out to be nonperturbative in low dimension and for an infinite-length polymer. In order to make progress, we redefine the task and analyze the wandering of a string of a finite length L. At zero temperature, we find that the displacement fluctuations $\langle [\delta u(X)]^2 \rangle \propto L X^2$ depend on L and scale with the square of the segment length X, which differs from a straightforward Larkin-type scaling. The result is best understood in terms of a typical squared angle $\langle \alpha^2 \rangle \propto L$, where $\alpha = \partial_x u$, from which the displacement scaling for the segment X follows naturally, $\langle [\delta u(X)]^2 \rangle \propto \langle \alpha^2 \rangle X^2$. At high temperatures, thermal fluctuations smear the disorder potential and the lowest-order results for disorder-induced fluctuations in both the displacement field and the angle vanish in the thermodynamic limit $L \to \infty$. The calculation up to the second order allows us to identify the regime of validity of the perturbative approach and provides a finite expression for the displacement correlator, albeit depending on the boundary conditions and the location relative to the boundaries.

1. INTRODUCTION

The $(d + n)$-dimensional random elastic manifold problem describes the behavior of the n-dimensional transverse displacement field $\mathbf{u}(\mathbf{r})$ of a d-dimensional elastic manifold subject to a disorder potential. The topic has been initiated with the 1970 work of Anatoli Larkin [1], where he analyzed how disorder distorts the three-dimensional vortex lattice in a type-II superconductor. Expanding the random potential $V[\mathbf{r}, \mathbf{u}(\mathbf{r})]$ due to impurities, he arrived at a random force model $V[\mathbf{r}, \mathbf{u}(\mathbf{r})] \approx \mathbf{f}(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r})$ and showed that the relative displacements of vortex positions $\delta \mathbf{u}(\mathbf{r})$ diverge with increasing distance R as $\langle [\delta u(R)]^2 \rangle^{1/2} \propto R^{3/2}$.

Later, the problem was generalized to manifolds of arbitrary dimensions $d + n$ and random potentials of other types, both long- and short-range correlated; the famous Larkin exponent $\zeta_L = (4 - d)/2$ describing the wandering $\langle [\delta u(R)]^2 \rangle^{1/2} \propto R^{\zeta_L}$ of an elastic manifold subject to a random force field is a tribute to the original analysis of Larkin. However, in low dimensions $d \leq 2$, the perturbative approach to the random elastic manifold problem breaks down at any scale and the problem becomes more difficult to solve.

Here, we study the short-scale behavior of an elastic string (or polymer) confined to a plane and subject to a disorder potential and thermal fluctuations, that is, the $(1 + 1)$-dimensional random directed polymer problem. The Hamiltonian describing the distortion $u(x)$ along the polymer involves two competing terms, the elastic energy H_0 (with J denoting the elasticity) and the potential energy $\mathcal{V}$ due to the random potential $V[x, u(x)]$:

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$$H = H_0 + \mathcal{V}, \quad H_0 = \frac{J}{2} \int\limits_0^L dx \, (\partial_x u)^2 \,,$$

$$\mathcal{V} = \int\limits_0^L dx \, V[x, u(x)] \,. \tag{1}$$

The random potential is usually assumed to be Gaussian distributed with zero mean and (short-ranged) correlated with a correlator $U(u)$ of width ξ (which may be related to the internal width of the manifold exposed to an underlying δ-correlated disorder potential),

$$\langle V(x, u) \rangle_d = 0 \,,$$
$$\langle V(x, u) V(x', u') \rangle_d = \delta(x - x') U(u - u') \,. \tag{2}$$

Here and below, angular brackets with a subscript "d" denote the average with respect to the distribution of the random potential V. The effect of thermal fluctuations (in a given realization of the random potential V) is described by the usual thermal average

$$\langle A \rangle_{th} = \frac{\mathrm{Sp}[A \exp(-\beta H)]}{\mathrm{Sp}[\exp(-\beta H)]} \,, \quad \beta \equiv 1/T \,, \tag{3}$$

and is identified by the subscript "th"; the average with a subscript "0" denotes the thermal average in the unperturbed system with the harmonic Hamiltonian H_0. The trace in Eq. (3) implies a functional integration over all trajectories $u(x)$ satisfying the chosen boundary conditions.

The generic question in the $1 + 1$-dimensional random directed polymer problem addresses the growth of the correlator

$$C_{d,th}(x_1, x_2) = \langle \langle [u(x_1) - u(x_2)]^2 \rangle_{th} \rangle_d \tag{4}$$

with the increase in $X = |x_2 - x_1|$, the distance between two points (with coordinates x_1 and x_2) of the polymer. In brief, the following understanding has been developed over the years: Starting from the perturbative analysis and the development of scaling arguments (Larkin scaling), it became clear (via numerical analysis [2], analytic work [3, 4], and a recent exact solution for $\xi = 0$ [5–7]) that the long-distance behavior is nonperturbative, with the wandering described in terms of a nontrivial wandering exponent ζ [8]

$$C_{d,th}(X) \sim \ell_\perp^2 \left(X/\ell_\parallel \right)^{2\zeta}, \quad X \gtrsim \ell_\parallel, \quad \zeta = 2/3 \,, \tag{5}$$

the transverse ($\ell_\perp$) and longitudinal ($\ell_\parallel$) scales being dependent on temperature. In the absence of disorder, the purely thermal wandering of the string is described by the correlator

$$C_0(x_1, x_2) = \langle (u_1 - u_2)^2 \rangle_0 \propto \frac{T}{J} X \tag{6}$$

(where we introduce the notation $u_i \equiv u(x_i)$) which exhibits a smaller wandering exponent $\zeta_{th} = 1/2$ than the one entering Eq. (5) and describing the disorder-induced wandering; hence, disorder-induced fluctuations supersede thermal wandering.

The breakdown of the perturbation theory at large scales is associated with the presence of many competing (disorder-induced) potential valleys. One could expect that only one valley is relevant at very short scales, which would make the perturbation theory applicable. This is indeed the case in higher dimensions $2 < d \leq 4$, but in low dimensions, particularly in $d = 1$, the perturbation theory also breaks down at small scales. For $T = 0$, this is easily understood from the lowest-order perturbative expression for $C_d(X)$, which corresponds to replacing the random potential by a random force $V[x, u(x)] \approx f(x) u(x)$ with the force $f(x)$ Gaussian distributed with zero mean and δ-correlated with the weight

$$\nu \equiv -U''(0) \tag{7}$$

given by the second derivative of the potential correlator $U(u)$. Within this approximation, the displacement correlator is given by the integral

$$C_d(X) = 2\nu \int d^d \mathbf{k} \, \frac{1 - \cos(\mathbf{k} \cdot \mathbf{X})}{(Jk^2)^2} \tag{8}$$

that diverges strongly at small k when $d \leq 2$. Conflicting results for $C_d(X)$ then have been quoted in the literature for $d = 1$, e. g., $C_d(X) \propto X^3$ resulting from the $d = 1$ Larkin scaling with $\zeta_L = (4 - d)/2 = 3/2$ (see Ref. [9]), or from the perturbative calculation for a polymer of a finite length X in the presence of a homogeneous random force [10]. A different result $C_d(X) \propto X^2$ was first obtained by Bouchaud et al. [11]; they analyzed the short-scale wandering of an infinite polymer using a self-consistent harmonic approximation in terms of a replica approach and accounting for replica symmetry breaking (see also the more recent work by Agoritsas et al. [12]). On the other hand, the replica theory does not provide the correct result for the wandering exponent on large scales, and there is an obvious need to check these findings with a better-controlled method. For this, we here analyze a slightly different problem, which is properly defined within a perturbative approach, namely, the fluctuations of a length-X segment of a polymer of length L, or, in other words, the fluctuations of a polymer in a box with properly defined boundary conditions (fixed or free). This finite-size-regularized perturbative approach does not rely on any uncontrolled approximations and provides interesting further insights into the problem.

In the remaining part of the introduction, we briefly summarize the results derived below and place them into the general context. For $T = 0$ and chosing a segment far away from the box ends (see Eq. (42) below), we find the displacement correlator

$$C_d(X \lesssim L) \sim (\nu/J^2) L\, X^2. \qquad (9)$$

This result depends on the box size L and diverges as $L \to \infty$; in addition, boundary effects show up when placing the segment close to the box ends. We note that for $X \sim L$, we recover the Larkin scaling at small distances. In analyzing the full random polymer problem, the result in Eq. (9) can be used up to lengths $L \sim L_c$, where the zero-temperature Larkin length

$$L_c \sim \left(\frac{J^2 \xi^2}{\nu} \right)^{1/3} \qquad (10)$$

can be obtained from dimensional estimates balancing the elastic energy $J\xi^2/L_c$ against the disorder energy $\sqrt{L_c \nu \xi^2}$. At $X \sim L \sim L_c$, we have $C_d(L_c) \sim \xi^2$ and the presence of other valleys becomes relevant. For $L \gtrsim L_c$, we cannot use the result (9) any more and have to replace the box size L with L_c, transforming (9) into

$$C_d(X \lesssim L_c) \sim \xi^2 \, (X/L_c)^2 \propto \nu^{2/3}, \qquad (11)$$

an explicitly nonperturbative result, as it is evident from the noninteger value of the exponent in the last relation. The scaling in Eq. (11), confirmed by our analysis below, was first reported in Ref. [11]. For large segments $X > L_c$, the wandering exponent is reduced to $\zeta = 2/3$ (cf. Eq. (5)). Hence, in the one-dimensional situation, the short-scale wandering at $u \ll \xi$ and $X \ll L_c$ explicitly depends on the larger scales ξ and L_c, a phenomenon known as intermittency in the field of turbulence [11].

To better understand the origin of the behavior $C_d \propto X^2$ in Eq. (11), we analyze the angle correlator $\langle \langle [\partial_x u\,(x_1) - \partial_x u\,(x_2)]^2 \rangle_{th} \rangle_d$, where the identification of the derivative $\partial_x u$ with the local angle α enclosed between the polymer and the x axis makes sense for sufficiently smooth excursions (directed polymer). This correlator does not suffer from the divergences appearing in the displacement correlator and hence produces a regular result (for $T = 0$)

$$\langle (\alpha_1 - \alpha_2)^2 \rangle_d \sim (\nu/J^2) X, \qquad (12)$$

where $\alpha_i \equiv \partial_x u|_{x=x_i}$. Relation (12) is easily derived from its perturbative expression having the form analogous to Eq. (8) but with an additional factor k^2 in the numerator. It is valid at scales $X \ll L_c$, where the perturbation theory holds; at larger distances $X \gtrsim L_c$,

the scaling $C_d(X) \sim \xi^2 [X/L_c]^{4/3}$ tells us that the angle correlator stops growing beyond L_c. The typical squared angle $\langle \alpha^2 \rangle_d = \langle [\partial_x u]^2 \rangle_d$ then assumes the value $(\nu/J^2) L_c \sim \xi^2/L_c^2$. As a result, the polymer wandering on large scales can be roughly visualized in terms of length-L_c segments, each enclosing a typical angle of the order of $\pm \xi/L_c$ with the x axis, from which it (super-)diffuses away with an exponent $\zeta > 1/2$. At the same time, we conclude that at small scales $X \ll L_c$, the displacement correlator grows as $C_d(X) \sim \langle \alpha^2 \rangle_d X^2 \sim (\nu/J^2) L_c X^2 \sim \xi^2 (X/L_c)^2$, providing an alternative derivation of the result in Eq. (11). The same arguments can be applied to derive the result in Eq. (9).

The analysis at a finite temperature $T > 0$ is more complex and involves the temperature-dependent Larkin length

$$L_c(T) \sim \frac{T^5}{U_0^2 J} \sim L_c \left(\frac{T}{T_{dpin}} \right)^5 \qquad (13)$$

with

$$U_0 = \int du\, U(u) \sim U(0)\xi, \qquad (14)$$

and the so-called thermal depinning temperature

$$T_{dpin} \sim (J U_0 \xi)^{1/3}. \qquad (15)$$

The temperature-dependent Larkin length $L_c(T)$ is the natural length scale that can be constructed from J, T, and U_0 (at high temperatures, the width of $U(u)$ cannot be resolved, and hence only its integral over u is relevant).

As shown in [13] (a more concise derivation can be found in the appendix of Ref. [14]), for a random potential uncorrelated in x, the full displacement correlator defined by Eq. (4) can be expressed as the sum of two terms,

$$C_{d,th}(x_1, x_2) = C_0(x_1, x_2) + C_d(x_1, x_2), \qquad (16)$$

the first of which, $C_0(x_1, x_2)$, has exactly the same form as in the absence of disorder (see Eq. (6)), whereas the second can be reduced to the form

$$C_d(x_1, x_2) = \langle (\langle u_1 - u_2 \rangle_{th})^2 \rangle_d. \qquad (17)$$

Accordingly, it is natural to call $C_d(x_1, x_2)$ the disorder-induced contribution to the fluctuations. At $T = 0$, this term includes only the disorder-induced fluctuations; at $T > 0$, these become subject to thermal smearing.

At $T > 0$, the main contribution to $C_{d,th}(x_1, x_2)$ at the smallest scales is always given by $C_0(x_1, x_2) \propto (T/J)X$ because the disorder-induced wandering is

always described by a larger exponent. As the distance X increases, the disorder-related term $C_d(x_1, x_2)$ overcomes the purely thermal contribution. We focus our attention entirely on the nontrivial disorder-induced contribution $C_d(x_1, x_2)$, whereas the consequences of the interplay between $C_0(x_1, x_2)$ and $C_d(x_1, x_2)$ are discussed in Ref. [12]. As is shown in Sec. 4.1 (see Eqs. (53) and (54)), the lowest-order result for the displacement correlator evaluated at a high temperature $T \gtrsim T_{dpin}$ scales as

$$C_d^{(1)}(X) \sim \langle \delta u^2[L_c(T)] \rangle_0 \sqrt{\frac{L_c(T)}{x_0}} \left(\frac{X}{L_c(T)} \right)^2 \quad (18)$$

and the estimate for the typical squared angle is

$$\langle \alpha^2(x_0) \rangle_d^{(1)} \sim \frac{\langle \delta u^2[L_c(T)] \rangle_0}{L_c^2(T)} \sqrt{\frac{L_c(T)}{x_0}}, \quad (19)$$

where x_0 marks the position of the segment X within the box $[0, L]$. The results in Eqs. (18) and (19) can be easily obtained from the corresponding $T = 0$ expressions by replacing the parameter $\nu = -U''$ with the ratio $U_0/\langle u^2(x_0) \rangle_0$, as motivated by the form of Eq. (33). Evaluating expression (18) away from the box ends (i. e., for $L - x_0 \sim x_0$) and taking the formal limit $L \to \infty$ (which corresponds to taking the thermodynamic limit), we see that these results vanish, in contrast to the zero-temperature results, which either increase with the box dimension L (as $C_d(X)$) or are independent thereof (as $\langle \alpha^2(x_0) \rangle_d$).

Evaluating the second-order term in the perturbation series (which, contrarily to the situation at $T = 0$, does not vanish), we obtain the result (cf. Eq. (57))

$$C_d^{(2)}(X) \sim f \left(\frac{x_0}{L} \right) \langle \delta u^2[L_c(T)] \rangle_0 \left(\frac{X}{L_c(T)} \right)^2. \quad (20)$$

Here, $f(x_0/L)$ is a smooth function of the relative location within the box (in our calculation below, we use fixed and free boundary conditions at $x = 0$ and $x = L$, respectively, resulting in a function $f(x_0/L)$ decreasing from a constant to zero). Hence, while the finite-temperature problem is less divergent than the zero-temperature version, it maintains some irregular features as the numerical factor in the result depends on the relative location of the segment X within the box.

Combining the first- and second-order results (18) and (20), we can conjecture that the perturbation expansion is valid as long as $x_0 \ll L_c(T)$. Pushing this result to its boundary of applicability, we find that

$$C_d^{(2)}(X) \sim \langle \delta u^2[L_c(T)] \rangle_0 \left(\frac{X}{L_c(T)} \right)^2, \quad (21)$$

the high-temperature analogue of Eq. (11), which smoothly goes over to the large-distance result (5) at $X \sim L_c(T)$. This result (as well as the $T = 0$ result in Eq. (11)) is then consistent with the findings in Ref. [12].

In Sec. 2 we construct the general form of the perturbative expansion for the model defined by Eqs. (1) and (2), which is applicable for any boundary condition. Section 3 is devoted to the discussion of the zero-temperature results for two different types of boundary conditions, and in Sec. 4 we analyze the situation at high temperatures.

2. CONSTRUCTION OF THE PERTURBATIVE EXPANSION

Expressing the disorder-induced correlator $C_d(x_1, x_2)$ in Eq. (17) through

$$c_d(x_1, x_2) \equiv \langle \langle u_1 \rangle_{th} \langle u_2 \rangle_{th} \rangle_d, \quad (22)$$

$$\begin{aligned} C_d(x_1, x_2) \equiv c_d(x_1, x_1) - c_d(x_1, x_2) - \\ - c_d(x_2, x_1) + c_d(x_2, x_2), \quad (23) \end{aligned}$$

we first have to calculate the thermal averages of $u_{1,2}$ in a given realization of a random potential. The first step towards this goal can be easily made by noting that the calculation of the thermal average of $\delta H/\delta u_1$ involves the integration of

$$(\delta H/\delta u_1) \exp(-\beta H) \equiv -T(\delta/\delta u_1) \exp(-\beta H)$$

over u_1, and therefore $\langle \delta H/\delta u_1 \rangle_{th}$ has to vanish. On the other hand, the variation of Eq. (1) with respect to u_1 gives

$$\frac{\delta H}{\delta u_1} = -J \, \partial_x^2 u \big|_{x=x'} + V_1', \quad (24)$$

where $V_1 \equiv V[x_1, u(x_1)]$ and the prime denotes the partial derivative with resect to u. Taking the thermal average of Eq. (24), we find that

$$\langle u_1 \rangle_{th} = - \int_0^L dx_2 \, g_{12} \langle V_2' \rangle_{th}, \quad (25)$$

where

$$g_{12} \equiv g(x_1, x_2) \quad (26)$$

is the response function of the unperturbed (harmonic) system, which is independent of temperature and is trivially related to the correlation function

$$c_0(x_1, x_2) \equiv \langle u_1 u_2 \rangle_0 \equiv G_{12} = T g_{12} \quad (27)$$

of the same system. Substituting Eq. (25) in Eq. (22) then gives

$$c_d(x_1, x_2) = \int\limits_0^L dx_3 \int\limits_0^L dx_4 \, g_{13}\Sigma_{34}g_{42}\,, \qquad (28)$$

where

$$\Sigma_{34} \equiv \langle\langle V_3'\rangle_{th}\langle V_4'\rangle_{th}\rangle_d \qquad (29)$$

plays the role of the effective self-energy part.

2.1. First-order approximation

In the first approximation, $\langle V_j'\rangle_{th}$ can be replaced by

$$\langle V_j'\rangle_0 = \left\langle \int\limits_{-\infty}^\infty \frac{dK_j}{2\pi} i K_j V(K_j) \exp(iK_j u_j) \right\rangle_0 =$$
$$= i\int\limits_{-\infty}^\infty \frac{dK_j}{2\pi} K_j V(K_j) w_{jj}^{1/2}\,, \quad (30)$$

where we perform the Fourier transformation with respect to u, introduce the notation

$$w_{ij} = \exp\left(-K_iK_jG_{ij}\right)\,, \qquad (31)$$

and take into account that $\langle u_j\rangle_0 = 0$. Note that in Eq. (30) (and analogous equations below), summation over repeated indices is not implied. Substituting Eq. (30) in Eq. (29) and averaging over disorder then provides the lowest-order approximation for the effective self-energy

$$\Sigma_{34}^{(1)} = \delta(x_3 - x_4)\sigma(x_3)\,,$$
$$\sigma(x_3) = \int\limits_{-\infty}^\infty \frac{dK_3}{2\pi} K_3^2 U(K)w_{33}\,. \qquad (32)$$

As $T \to 0$, thermal fluctuations are suppressed such that $G_{33} \to 0$ and $w_{33} \to 1$. In this limit, the expression for $\sigma(x)$ in Eq. (32) is reduced to $\sigma = -U''(u = 0) = $ $= $ const (in agreement with the zero-temperature analysis of Efetov and Larkin [15]). On the other hand, at $T > 0$, we have $w_{33} < 1$ and hence the value of $\sigma(x_3)$ is suppressed in comparison with its zero-temperature value and depends on x_3. In particular, for $U(u) \propto \exp(-x^2/4\xi^2)$, that is, for

$$U(K) = U_0 \exp(-\xi^2 K^2),$$

the integration over K_3 in Eq. (32) is Gaussian and can be performed exactly, which leads to

$$\sigma(x_3) = \frac{U_0}{4[\pi(\xi^2 + G_{33})^3]^{1/2}}\,. \qquad (33)$$

This suggests that at high temperatures (corresponding to $G_{33} \gg \xi^2$), the random potential is smeared out by thermal fluctuations in such a way that its correlation radius ξ is replaced by the typical thermal displacement $G_{33}^{1/2} = \langle u_3^2\rangle_0^{1/2}$, although $U_0 \equiv \int du\, U(u)$ remains the same (cf. with Ref. [9]).

2.2. Second-order approximation

To find the next nonvanishing contribution to Σ_{34}, we have to calculate $\langle V_j'\rangle_{th}$ with the help of Eq. (3), expanding the exponentials both in the nominator and in the denominator in powers of $\mathcal{V}$ up to the second order,

$$\langle V_j'\rangle_{th} = \frac{\langle V_j'\rangle_0 - \beta\langle V_j'\mathcal{V}\rangle_0 + (\beta^2/2)\langle V_j'\mathcal{V}^2\rangle_0 + \cdots}{1 - \beta\langle\mathcal{V}\rangle_0 + (\beta^2/2)\langle\mathcal{V}^2\rangle_0 + \cdots} =$$
$$= \langle V_j'\rangle_0 - \beta\left[\langle V_j'\mathcal{V}\rangle_0 - \langle V_j'\rangle_0\langle\mathcal{V}\rangle_0\right] +$$
$$+ \frac{\beta^2}{2}\left[\langle V_j'\mathcal{V}^2\rangle_0 - 2\langle V_j'\mathcal{V}\rangle_0\langle\mathcal{V}\rangle_0 +\right.$$
$$\left. + 2\langle V_j'\rangle_0\langle\mathcal{V}\rangle_0^2 - \langle V_j'\rangle_0\langle\mathcal{V}^2\rangle_0\right] + \cdots \quad (34)$$

The contribution to the product $\langle V_3'\rangle_{th}\langle V_4'\rangle_{th}$ of the second order in powers of $\mathcal{V}$ can then be written as

$$[\langle V_3'\rangle_{th}\langle V_4'\rangle_{th}]^{(2)} = \beta^2\mathrm{sym}_{34}\left[\langle V_3'\mathcal{V}^2\rangle_0\langle V_4'\rangle_0 -\right.$$
$$- 4\langle V_3'\mathcal{V}\rangle_0\langle\mathcal{V}\rangle_0\langle V_4'\rangle_0 + 3\langle V_3'\rangle_0\langle\mathcal{V}\rangle_0^2\langle V_4'\rangle_0 -$$
$$\left. - \langle V_3'\rangle_0\langle\mathcal{V}^2\rangle_0\langle V_4'\rangle_0 + \langle V_3'\mathcal{V}\rangle_0\langle\mathcal{V}V_4'\rangle_0\right]\,, \quad (35)$$

where $\mathrm{sym}_{34}[\dots]$ implies taking only the part of the expression in square brackets that is symmetric with respect to the permutation of x_3 and x_4.

Taking the disorder average of Eq. (35) produces the following expression for the second-order contribution to Σ_{34}:

$$\Sigma_{34}^{(2)} = \beta^2 \int\limits_{-\infty}^\infty \int\limits_{-\infty}^\infty \frac{dK_3}{2\pi}\frac{dK_4}{2\pi} U(K_3)U(K_4) \times$$
$$\times \left[\delta(x_3 - x_4)\delta(K_4 - K_5)K_3^2 \times \right.$$
$$\times \int\limits_0^L dx_5 w_{33}w_{55}(3 - 4w_{35} + w_{35}^2) -$$
$$\left. - K_3K_4w_{33}w_{44}(2w_{34} - w_{34}^2)\right]\,. \quad (36)$$

In the expansion of $(3 - 4w_{35} + w_{35}^2)$ in powers of $K_3 K_5 G_{35}$ the first nonvanishing term after the integration over K_3 and K_4 is proportional to T^4, whereas the expansion of $(2w_{34} - w_{34}^2)$ and integration over K_3 and K_4 produces a first nonvanishing term proportional to T^3. This ensures that in the limit $T \to 0$, $\Sigma_{34}^{(2)}$ vanishes, in agreement with the zero-temperature analysis of Efetov and Larkin [15].

For $U(K) = U_0 \exp(-\xi^2 K^2)$, the integration over K_3 and K_4 in Eq. (36) is Gaussian and can be performed exactly. The result of this integration can be expressed in the relatively compact form

$$\Sigma_{34}^{(2)} = \frac{U_0^2}{8\pi T^2} \left[\delta(x_3 - x_4) \int_0^L dx_5 \, (G_{55} + \xi^2) \times \right.$$
$$\left. \times \left(3D_{35}^{(0)} - 4D_{35}^{(1)} + D_{35}^{(2)} \right) + G_{34} \left(D_{34}^{(1)} - D_{34}^{(2)} \right) \right],$$

where

$$D_{ij}^{(p)} = [(G_{ii} + \xi^2)(G_{jj} + \xi^2) - (p/2)^2 G_{ij}^2]^{-3/2}, \quad (37)$$
$$p = 0, 1, 2.$$

3. ZERO TEMPERATURE

3.1. Fixed boundary conditions

We first consider the case where both end points of the string are fixed, $u(0) = u(L) = 0$. In this situation,

$$g_{12} = \frac{\bar{x}_1 (L - \bar{x}_2)}{LJ}, \quad (38)$$

where

$$\bar{x}_1 = \min(x_1, x_2), \quad \bar{x}_2 = \max(x_1, x_2),$$

and hence $\bar{x}_1 \le \bar{x}_2$. As demonstrated in Ref. [15], only the lowest-order contribution to the perturbative expansion of Σ is nonzero at $T = 0$,

$$\Sigma_{34} = \delta(x_3 - x_4)[-U''(0)]. \quad (39)$$

Substituting Eqs. (38) and (39) in Eq. (28) and integrating over x_3 leads to

$$c_d(x_1, x_2) = -\frac{U''(0)}{6J^2 L}[2L\bar{x}_2 - x_1^2 - x_2^2]\,\bar{x}_1\,(L - \bar{x}_2), \quad (40)$$

which, with Eq. (23), allows expressing $C_d(x_1, x_2)$ as

$$C_d(x_1, x_2) = \frac{-U''(0)}{3J^2 L} \times$$
$$\times \left[\bar{x}_1^2 + (L - \bar{x}_2)^2 - \bar{x}_1 (L - \bar{x}_2) \right] (x_1 - x_2)^2. \quad (41)$$

Equation (41) demonstrates that even for $L \gg X$, C_d essentially depends on the total system size L,

$$C_d \sim \frac{-U''(0)}{J^2} L\, X^2, \quad (42)$$

and hence the thermodynamic limit $L \to \infty$ cannot be taken. This property was discovered by Bouchaud, Mezard and Parisi [11], who noticed that in an infinite system, the first-order expression for the correlator $C_d(X)$ is given by an integral (see Eq. (8)) that for $d = 1$ diverges at small k as $1/k_{min}$. Equation (41) confirms this conclusion on a more quantitative level and demonstrates that the correlator C_d remains proportional to L even when the distance from the interval X where the relative displacement to the boundary of the system is measured is much smaller than L.

According to Eq. (41), for $X \ll L$, the dependence of $C_d(X)$ on X reduces to $C_d(X) \propto X^2$. This means that the main contribution to the fluctuations of the relative displacement in such a regime comes from the fluctuations of the derivative $\alpha(x) = \partial_x u$, which only weakly changes between x_1 and x_2. For $|\alpha(x)| \ll 1$, this derivative coincides with the angle between the polymer and its average direction (along axis x), and we therefore call the variable $\alpha(x)$ the angle in what follows. We note that Hamiltonian (1) is directly applicable only when the values of $|\alpha(x)|$ are much smaller than 1. If the parameters of the system are such that typical fluctuations of α calculated with the help of Hamiltonian (1) are much larger than unity, this Hamiltonian has to be complemented by some terms producing a short-scale cutoff that suppresses the fluctuations of the angle; for example, a term proportional to $(\partial_x^2 u)^2$ can be included into the elastic part.

In terms of angle fluctuations, Eq. (42) can be rewritten as

$$C_d(X) = \langle \alpha_0^2 \rangle_d X^2, \quad (43)$$

where $\alpha_0 \equiv \alpha(x_0)$ and $x_0 \approx x_{1,2}$ (more precisely, x_0 lies between x_1 and x_2). It follows from Eq. (28) that for $\Sigma_{34} = \delta(x_3 - x_4)\sigma(x_3)$, the expression for $\langle \alpha_1 \alpha_2 \rangle_d$ is

$$\langle \alpha_1 \alpha_2 \rangle_d = \int_0^L dx_3 \, \frac{\partial g_{13}}{\partial x_1} \frac{\partial g_{23}}{\partial x_2} \, \sigma(x_3), \quad (44)$$

whence (at $T = 0$),

$$\langle \alpha_1 \alpha_2 \rangle_d = \frac{-U''(0)}{2J^2 L} \left[\bar{x}_1^2 + (L - \bar{x}_2)^2 - L^2/3 \right] \quad (45)$$

and

$$\langle \alpha^2(x) \rangle_d = \frac{-U''(0)}{J^2} L \left[\frac{1}{12} + \left(\frac{x}{L} - \frac{1}{2} \right)^2 \right]. \quad (46)$$

Equation (46) shows that $\langle \alpha^2(x) \rangle_d$ is proportional to the total length L of the string, implying that its value in a given realization of the random potential is determined by the fluctuations of the random potential at large length scales (which are comparable with L) and not by those in the close vicinity of the point x. The value of $\langle \alpha^2(x) \rangle_d$ is minimal in the middle of the string and is larger at its endpoints by a factor of 4.

To find the range of X where C_d is determined by angle fluctuations and is therefore proportional to X^2, we have to compare

$$\langle (\alpha_1 - \alpha_2)^2 \rangle_d = \frac{-U''(0)}{J^2} |x_1 - x_2| \qquad (47)$$

with the value of $\langle \alpha^2(x) \rangle_d$ given by Eq. (46). This comparison confirms that the dependence $C_d \propto X^2$ holds as long as $X \ll L$.

3.2. Mixed boundary conditions

If only one endpoint of the string is fixed and the other is free, the equations become slightly simpler. In particular, if the boundary condition at $x = 0$ is $u(0) = 0$ and free at $x = L$, then Eq. (38) is replaced by

$$g_{12} = \bar{x}_1 / J . \qquad (48)$$

Quite remarkably, for such boundary conditions, the form of the Green's function $g_{12} \equiv g(x_1, x_2)$ is sensitive only to one of its two arguments and does not depend on the total length L. In this situation, the results in Eqs. (41) and (46) are respectively replaced by

$$C_d(x_1, x_2) = -\frac{U''(0)}{J^2} \left(L - \frac{\bar{x}_1 + 2\bar{x}_2}{3} \right) (x_1 - x_2)^2, \quad (49)$$

and

$$\langle \alpha^2(x) \rangle_d = \frac{-U''(0)}{J^2} (L - x_0) . \qquad (50)$$

It follows from the form of Eq. (49) that for $T = 0$ and mixed boundary conditions, C_d depends only on the distances $y_{1,2} = L - x_{1,2}$ between the points $x_{1,2}$ and the endpoint with the free boundary condition, whereas the total length of the string is of no importance. In particular, when the points x_1 and x_2 are situated in the vicinity of the free endpoint of the string, C_d is proportional to the distance from the free end and not to the total length L.

Naturally, an analogous conclusion applies also to $\langle \alpha^2(x) \rangle_d$, which linearly depends on $L - x$. The reason why $\langle \alpha^2(x) \rangle_d$ vanishes as $x \to L$ is quite clear: in the minimal-energy configuration, the value of $\alpha(x)$ at the

free end ($x = L$) has to be equal to zero for any realization of the random potential. This follows from the variation of Hamiltonian (1) with respect to $u(x = L)$.

For a Green's function g_{12} of form (48), differentiating Eq. (28) with $\Sigma_{34} = \delta(x_3 - x_4)\sigma(x_3)$ gives

$$\langle \langle \alpha_1 \rangle_{th} \langle \alpha_2 \rangle_{th} \rangle_d = \langle \langle \alpha(\bar{x}_2) \rangle_{th}^2 \rangle_d \qquad (51)$$

whence it follows that the zero-temperature expression for $\langle (\alpha_1 - \alpha_2)^2 \rangle_d$ has exactly the same form (given by Eq. (47)) as in the case of fixed boundary conditions. A comparison of Eq. (47) with Eq. (50) then suggests that for mixed boundary conditions, the dependence $C_d \propto X^2$ holds as long as X is much smaller than the distance to the free end of the string.

4. FINITE TEMPERATURES

4.1. First-order approximation, mixed boundary conditions

At finite temperatures, all terms of the perturbative expansion of Σ become nonzero, and we start by analyzing the first term. For simplicity, we consider the case of mixed boundary conditions and focus our attention on sufficiently high temperatures, where we can neglect the finiteness of ξ, that is, $U(u)$ can be replaced by $U_0\delta(u)$. Substituting $G_{33} = (T/J)x_3$ in Eq. (33) with $\xi = 0$ shows that in this regime, $\sigma(x)$ depends algebraically on the distance from the fixed end of the string,

$$\sigma(x) = \frac{U_0}{4\sqrt{\pi}} \left(\frac{J}{Tx} \right)^{3/2} . \qquad (52)$$

Substituting $\Sigma_{34} = \delta(x_3 - x_4)\sigma(x_3)$ in Eqs. (28) and (44) and integrating over x_3 then lead to

$$C_d^{(1)}(x_1, x_2) = \frac{U_0}{2(\pi J T^3)^{1/2}} \times$$

$$\times \left[\frac{4}{3} \frac{2\bar{x}_1^{1/2} + \bar{x}_2^{1/2}}{\left(x_1^{1/2} + x_2^{1/2} \right)^2} - \frac{1}{L^{1/2}} \right] (x_1 - x_2)^2 \quad (53)$$

and

$$\langle \langle \alpha(x) \rangle_{th}^2 \rangle_d^{(1)} = \frac{U_0}{2(\pi J T^3)^{1/2}} \left(\frac{1}{x^{1/2}} - \frac{1}{L^{1/2}} \right) . \quad (54)$$

The result in Eq. (53) implies that at $T > 0$ (analogously to what is the case at $T = 0$), the disorder-induced fluctuations of the thermally averaged relative displacement $\langle u_1 - u_2 \rangle_{th}$ at small X are determined by fluctuations of the thermally averaged angle $\langle \alpha \rangle_{th}$.

In a generic situation with $x_{1,2} \sim L - x_{1,2} \sim L$, expression (53) providing the first-order contribution to $C_{d,th}(X)$ vanishes in the limit $L \to \infty$ (in contrast to the expectations expressed in Sec. 2.2 in Ref. [9]) and remains finite only if the segment X is situated at a finite distance from the fixed endpoint of the string, where the influence of thermal fluctuations is not as pronounced as in the middle of an infinite string. Naturally, the same holds for expression (54) for $\langle\langle\alpha(x)\rangle_{th}^2\rangle_d^{(1)}$, the intensity of fluctuations of the thermally averaged angle.

For finite L, on approaching the free end of the string, the angle fluctuations $\langle\langle\alpha(x)\rangle_{th}^2\rangle_d^{(1)}$ tend to zero linearly in the distance $L - x$, exactly as is the case with $\langle\alpha^2(x)\rangle_d^{(1)}$ at $T = 0$ (see Eq. (46)). However, the reasons for this result are less evident than at $T = 0$, because in the presence of thermal fluctuations, $\alpha(x = L)$ is not obliged to be equal to zero. On the other hand, the reasons for the divergence of $\langle\langle\alpha(x)\rangle_{th}^2\rangle_d^{(1)}$ with decreasing x are quite clear: the amplitude of thermal fluctuations of the displacement decreases on approaching the fixed end of the string, and therefore the thermal suppression of angle fluctuations also becomes less prominent. Naturally, in a system with $\xi > 0$, dependence (54) has to saturate at sufficiently small x because $\langle\langle\alpha(x)\rangle_{th}^2\rangle_d^{(1)}$ cannot exceed its zero-temperature limit $\langle\alpha^2(x)\rangle_d^{(1)}$.

It follows from Eqs. (51) and (54) that the angle correlator at high temperatures assumes the form

$$\langle\langle\alpha_1 - \alpha_2\rangle_{th}^2\rangle_d^{(1)} =$$
$$= \frac{U_0}{2(\pi J T^3)^{1/2}} \frac{X}{(x_1 x_2)^{1/2}(x_1^{1/2} + x_2^{1/2})} . \quad (55)$$

The comparison of Eqs. (54) and (55) allows concluding that for the applicability of the dependence $C_d(X) \propto X^2$ (which requires having $\langle\langle\alpha_1 - \alpha_2\rangle_{th}^2\rangle_d \ll \langle\langle\alpha\rangle_{th}^2\rangle_d$), the length X of the segment has to be much smaller than the distance to the nearest endpoint.

4.2. Second-order approximation, mixed boundary conditions

To find the range of parameters where the value of $a_d(x_0)$ can be described by expression (54), we have to consider the next term in the perturbative expansion. Substituting Eq. (37) in

$$\langle\langle\alpha_0\rangle_{th}^2\rangle_d = \int\limits_0^L \int\limits_0^L dx_3 \, dx_4 \, \frac{\partial g_{03}}{\partial x_0} \, \frac{\partial g_{04}}{\partial x_0} \Sigma_{34} \quad (56)$$

and subsequently integrating over x_3 and x_4 leads to

$$\langle\langle\alpha(x_0)\rangle_{th}^2\rangle_d^{(2)} = \frac{U_0^2}{2\pi T^4} f\left(\frac{x_0}{L}\right) , \quad (57)$$

where

$$f(\gamma) = 3\left(\gamma^{-1/2} - 1\right) - 2\left(\sqrt{4\gamma^{-1} - 1} - \sqrt{3}\right) + $$
$$+ \sqrt{\gamma^{-1} - 1} . \quad (58)$$

The function $f(\gamma)$ has a finite limit $f(\gamma \to 0) = 2\sqrt{3} - 3 \approx 0.464$ and monotonically decreases with increasing γ. The behavior of $f(\gamma)$ on approaching $\gamma = 1$ is determined by the last term in Eq. (58), $f(\gamma \to 1) \approx (1 - \gamma)^{1/2}$.

For $L - x_0 \sim L$, that is away from the free boundary, Eqs. (54) and (57) can be rewritten as

$$\langle\langle\alpha(x_0)\rangle_{th}^2\rangle_d^{(1)} \sim \frac{U_0^2}{T^4}\left[\frac{L_c(T)}{x_0}\right]^{1/2} \sim$$
$$\sim \frac{T}{J L_c(T)}\left[\frac{L_c(T)}{x_0}\right]^{1/2} , \quad (59)$$

$$\langle\langle\alpha(x_0)\rangle_{th}^2\rangle_d^{(2)} \sim \frac{U_0^2}{T^4} \sim \frac{T}{J L_c(T)} , \quad (60)$$

which suggest that the perturbative expansion for $a_d(x_0)$ (and, therefore, for $C_d \approx \langle\langle\alpha\rangle_{th}^2\rangle_d X^2$) is in powers of $[x_0/L_c(T)]^{1/2}$ and, accordingly, the condition for its applicability is $x_0 \ll L_c(T)$. This seems to be reasonable: if we expect that the condition for the applicability of the perturbation theory at $T = 0$ is $x_0 \ll L_c \sim (J^2\xi^5/U_0)^{1/3}$, then it is rather natural that at large temperatures, L_c is replaced by its high-temperature analog $L_c(T)$. Pushing the result to the boundary of its validity range (i.e., evaluating Eqs. (59) and (60) at $x_0 \sim L_c(T)$), we obtain the result $C_d(X) \sim (U_0^2/T^4)X^2$; the condition $C_d[L_c(T)] \sim \langle u^2[L_c(T)]\rangle_0$ leads to expression (13) for the temperature-dependent Larkin length and the condition $\langle u^2[L_c(T_{dpin})]\rangle_0 \sim \xi^2$ leads to the depinning temperature in (15). Rewriting $C_d(X)$ in terms of these quantities, we obtain the result in Eq. (21).

As regards the vicinity of the other (free) boundary, the situation is less clear. In this region, the first-order term $\langle\langle\alpha\rangle_{th}^2\rangle_d^{(1)} \propto (L - x_0)$ vanishes more rapidly than the second-order contribution $\langle\langle\alpha\rangle_{th}^2\rangle_d^{(2)} \propto \sqrt{L - x_0}$ and hence the ratio of the second- and the first-order terms is of the order of $L/[(L - x_0)L_c(T)]^{1/2}$; this suggests that on approaching the free endpoint, one may always leave the applicability range of the perturbation theory.

5. CONCLUSION

We have analyzed the wandering of a directed polymer in a random potential. Formulated for an infinite polymer, this problem is nonperturbative at all scales. For a sufficiently small finite system, the problem is perturbative with the polymer excursions depending on the system size, the boundary conditions, and the relative position of the segment within the box. The results for the infinite polymer at distances $X \ll L_c(T)$ can be found from the finite-system results by replacing the box size L with the Larkin scale $L_c(T)$. The physical interpretation of the results is the following: at short scales, the polymer fluctuates around an almost straight line, which, however, deviates from the original direction or the x axis by some angle $\alpha = \partial_x u$. The typical value of this angle depends on the box size L and on the relative location x_0/L in the box; for the infinite polymer, we have to replace $x_0 \sim L \sim L_c$. As a result, the wandering at short scales is close to linear, $\delta u \sim \alpha X$, but with a coefficient α the typical value of which is determined by much larger scales.

The linear behavior of the displacement, or $C_d(X) \approx B\xi^2(X/L_c)^2$ including the numerical factor B, was previously obtained in [11] and [12] in the framework of a variational replica ansatz incorporating replica symmetry breaking. Our analysis demonstrates that their approach correctly captures the scaling $\propto X^2$ of the fluctuations at small scales (in contrast to large scales $X \gtrsim L_c$, where this approach does not predict the correct result for the wandering exponent) and provides a simple physical interpretation for this behavior in term of angle fluctuations. On the other hand, an accurate calulation of the coefficient B in the expression for $C_d(X)$ will most likely require the use of more advanced methods than the variational one.

In Ref. [9], the short-distance behavior of the disorder-induced correlator $C_d(X)$ for a polymer of infinite length was analyzed for high temperatures $T \gg T_{dpin}$ in the framework of a dynamic analysis based on the Langevin equation. In this scheme, the total displacement $u(x,t)$ is split into a thermal and a disorder-induced parts, $u(x,t) = u_{th}(x,t) + u_d(x,t)$, with the two terms following from the iterative solution of the Langevin equation

$$[\eta\partial_t - J\partial_x^2]\, u(x,t) = f_{th}(x,t) + f_d[x, u(x,t)] \qquad (61)$$

with a δ-correlated thermal force $\langle f_{th}(x,t)f_{th}(x',t')\rangle =$ $= 2\eta T\delta(x - x')\,\delta(t - t')$, where η is the friction coefficient for the dissipative motion and $f_d[x, u(x,t)] =$ $= -\partial_u V[x, u(x,t)]$ is the disorder-induced force. By expanding in u_d, the term $\langle\langle[\delta u_d(X)]^2\rangle_{th}\rangle_d$ provided the result $C_d(X) \sim \langle\delta u^2[L_c(T)]\rangle_0\, [X/L_c(T)]^{3/2}$ consistent

with the Larkin scaling. But this result is wrong because this term is exactly compensated [16] by the term $2\langle\langle\delta u_d(X)\,\delta u_{th}(X)\rangle_{th}\rangle_d$ that was missed in the analysis in Ref. [9]. This vanishing of the first-order term in the perturbative analysis of the high-temperature analysis is in agreement with the result in Eq. (53) when evaluating the expression away from the box ends and taking the formal limit $L \to \infty$ for the box size.

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