XY model on a Sierpinski gasket

A. Vallat
Institut de Physique, Université Neuchâtel, Breguet 1, CH-2000 Neuchâtel, Switzerland

S. E. Korshunov
L. D. Landau Institute for Theoretical Physics, Kosygina, 2, 117334 Moscow, USSR

H. Beck
Institut de Physique, Université de Neuchâtel, Breguet 1, CH-2000 Neuchâtel, Switzerland

(Received 6 August 1990)

Correlation functions and topological excitations of the XY model on a Sierpinski gasket are studied. The energy of a vortex is shown to be finite, so no Berezinskii-Kosterlitz-Thouless transition can be expected to take place. At any temperature the correlation function decays exponentially at large distances. A form of the XY model on a Sierpinski gasket is found that allows for exact renormalization. The results obtained can be applied to superconducting wire networks and tunnel-junction arrays.

I. INTRODUCTION

Recent progress in the techniques of lithography has led to the development of an interesting branch of experimental physics, namely, the fabrication and investigation of two-dimensional superconducting systems (tunnel-junction arrays and wire networks) with different regular or irregular structures.[1] This means that one has the possibility to produce quasicrystalline, fractal, or some other nontrivial superconductors. For example, wire networks in the form of a Sierpinski gasket have been studied by Gordon *et al.* [2] and more recently by Martinoli *et al.* [3]

On the other hand theoretical investigations of fractal superconducting wire networks have not, so far, gone beyond the application of the Landau-Ginzburg approximation.[4,5] Such an approach is certainly not sufficient, since even in two dimensions the phase fluctuations of the superconducting order parameter are very important both for the properties of the ordered phase and for the phase transition. In the case of a Sierpinski gasket with effective dimensionality less than two one can expect them to be of even greater importance.

In this paper we investigate the influence of the fluctuations on the properties of a fractal superconducting system on the example of the Sierpinski gasket (Fig. 1). When only the most relevant phase fluctuations are taken into account, the 2D-superconducting system can be described by one or another type of XY model. The Josephson junction array, for example, will be described by the ordinary XY model with cosine interaction

$$H=-J\sum_{\langle \mathbf{r},\mathbf{r}'\rangle}\cos(\theta_{\mathbf{r}}-\theta_{\mathbf{r}'})\ , \tag{1}$$

where variables θ_r's stand for the phases of the order parameter of corresponding superconducting grains. The summation in Eq. (1) is taken over pairs of nearest neighbors on a lattice that are connected by Josephson junctions.

On the other hand the phase fluctuations in the wire network should be described by the Hamiltonian

$$H=\int dl\frac{J_0}{2}\left[\frac{\partial\theta}{\partial l}\right]^2, \tag{2}$$

where integration should be performed over the whole network bearing in mind that θ is a multivalued function. Hamiltonian (2) being quadratic in θ, one can integrate out from the corresponding partition function the fluctuations of θ on the wires but not on the nodes of the network, obtaining in such a way

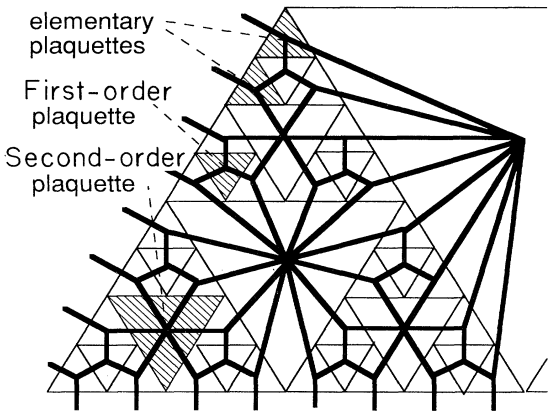

FIG. 1. The Sierpinski gasket is constructed with elementary triangles (or elementary plaquettes). The first-order fractal is formed by combining three of them. In the general case the $(s+1)$-order fractal is obtained by the juxtaposition of three s-order fractals. We will call the greatest plaquette which is situated in the center of s-order fractal an s-order plaquette, and sites at its three corners the s-order sites. The dual lattice is shown in bold lines. Each bond of a dual lattice intersects a bond of a direct lattice.

$$Z=\sum_{\{m_{rr'}\}}\exp\left[-\frac{K}{2}\sum_{\langle r,r'\rangle}(\theta_{r'}-\theta_r-2\pi m_{rr'})^2\right],$$
$$K=\frac{J}{k_BT},\quad J=\frac{J_0}{L}, \tag{3}$$

where θ_r stands for the values of θ at the nodes of the network and m is the winding number. In the general case J would be dependent on the length L of the link, but we will be interested only in the case of all links being of equal length. The partition function of the form (3) corresponds to an interaction between θ_r and $\theta_{r'}$ that is very close to a cosine function for $K\ll 1$ and to a piecewise parabolic function for $K\gg 1$. The interaction incorporated in Eq. (3) was initially introduced by Berezinskii[6] and Villain[7] instead of a cosine for the sake of convenience, because it allows for a number of various rigorous transformations to other representations. We would like to emphasize here that for a network (in contrast to an array) this particular form of the interaction is the most adequate.

We start our investigation of the XY model on a Sierpinski gasket by developing the harmonic approximation in Sec. II. Then we turn to the investigation of topological excitations (vortices) in Sec. III and finally find a kind of description that explicitly incorporates the periodicity of the phase in Sec. IV. Our main conclusion is that in such models there is no phase transition in the rigorous sense of the word. For arbitrarily low temperature the energy of the vortices is finite and the correlation functions decay exponentially at large enough distances. So in such a system there will be no global phase coherence.

II. HARMONIC APPROXIMATION

In the harmonic approximation any XY model will be described by the Hamiltonian

$$H=\frac{J}{2}\sum_{\langle rr'\rangle}(\theta_r-\theta_{r'})^2=\frac{k_BT}{2}\sum_{r,r'}\theta_r U_{rr'}\theta_{r'}, \tag{4}$$

where the elements of the dimensionless interaction matrix $U_{rr'}$ are equal to $4K$ for $\mathbf{r}=\mathbf{r}'$, to $-K$ for $\mathbf{r}$ and $\mathbf{r}'$ being the nearest neighbors and to 0 elsewhere. The correlation function $g_{rr'}$ will then be determined by the reciprocal matrix $G_{rr'}=(U^{-1})_{rr'}$:

$$\begin{aligned}g_{rr'}&\equiv\langle\exp[i(\theta_r-\theta_{r'})]\rangle\\&=\exp[-\tfrac{1}{2}\langle(\theta_r-\theta_{r'})^2\rangle]\\&=\exp(-\tfrac{1}{2}F_{rr'})\end{aligned} \tag{5}$$

with

$$F_{rr'}\equiv G_{rr}-G_{rr'}-G_{r'r}+G_{r'r'}. \tag{6}$$

$G_{rr'}$ is proportional to the lattice Green function of the Sierpinski gasket. To calculate $G_{rr'}$, it is convenient to express it as

$$G_{rr'}=-\left[\frac{\partial}{\partial h_r}\frac{\partial}{\partial h_{r'}}\ln Z\{h\}\right]_{h=0}, \tag{7}$$

where $Z\{h\}$ stands for the generating function

$$Z\{h\}\equiv\prod_r\left[\int_{-\infty}^{\infty}\frac{d\theta_r}{2\pi}\right]\exp\left[-\frac{K}{2}\sum_{\langle r,r\rangle}(\theta_r-\theta_{r'})^2+i\sum_r h_r\theta_r\right] \tag{8}$$

which for $h=0$ coincides with the partition function Z.

The fractal structure of the Sierpinski gasket allows us to calculate $Z\{h\}$ (and consequently $G_{rr'}$) step by step, by integrating out to the sth step the variables that are defined on the corner sites of the sth order plaquettes of the initial lattice (we shall call such sites the sth order sites, see Fig. 1). For example, after the first integration one obtains

$$Z\{h\}=A_1V_1\{h\}Z_1\{h\}, \tag{9}$$

where A_1 is the numerical factor depending on K, which is irrelevant for calculating G, and $Z_1\{h\}$ has the same structure as $Z\{h\}$:

$$Z_1\{h\}=\prod_r\left[\int_{-\infty}^{\infty}\frac{d\theta_r}{2\pi}\right]\exp\left[-\frac{K_1}{2}\sum_{\langle rr'\rangle}(\theta_r-\theta_{r'})^2+i\sum_r h'_r\theta_r\right] \tag{10}$$

but with a renormalized value of the coupling constant $K_1=(\frac{3}{5})K$ and the variables defined on the decimated lattice. Due to the property of self-similarity, the decimated lattice is again the Sierpinski gasket. In the process of integration, each field h_r defined on the first-order sites splits into three nonequal parts, which are shifted to the neighboring sites of larger order (see Fig. 2), so h' in Eq. (10) stands for the

$$h_r+\sum_{r'}\gamma_{rr'}h_{r'}, \tag{11}$$

where $\gamma_{rr'}=\frac{1}{5}$ or $\frac{2}{5}$ and the sum is taken over six nearest first-order sites.

The exponent of the factor

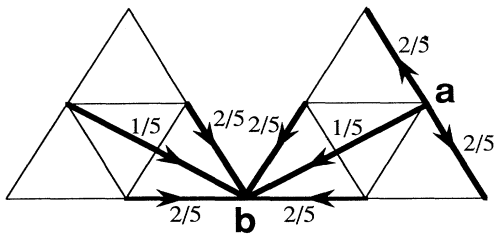

FIG. 2. Illustration of the "splitting" of h_a at the first-order site a. At the superior order site b, all contributions from all first-order sites to h_b are shown.

$$V_1\{h\}=\exp\left[-\frac{1}{20K}\sum(h_{r_1}^2+h_{r_2}^2+h_{r_3}^2+3h_{r_1}h_{r_2}+3h_{r_2}h_{r_3}+3h_{r_3}h_{r_1})\right], \tag{12}$$

where the summation is to be taken over all first order plaquettes and $\mathbf{r}_\alpha$ ($\alpha=1,2,3$) are the sites at their corners, gives the first contribution to $G_{rr'}$.

One can repeat the decimation procedure again and again. At each step the structure of $Z_s\{h\}$ is conserved, but the coupling constant is scaled by the factor of $\frac{3}{5}$ and a new additive contribution to $G_{rr'}$ appears of the same form as the exponent in Eq. (12). The series for $G_{rr'}$ turns out to be divergent, but for the combination (6) entering the correlation function (5), it either has a finite number of terms or it is convergent. Unfortunately it is impossible to sum this series analytically.

One can understand how the correlation function decays by considering it for, say, sites $\mathbf{r}'$ and $\mathbf{r}''$ which are of the s'th and s'th orders, respectively. Thus one can perform the renormalization procedure for $s=\min\{s',s''\}-1$ times, obtaining the expression for the correlation function for the model [see Eqs. (5)–(8)] with $K_R=(\frac{3}{5})^s K$ with distance r between points equal to $(\frac{1}{2})^s|\mathbf{r}'-\mathbf{r}''|$. This shows that the correlation function $g(\mathbf{r})$ decays as

$$g(r)\propto\exp\left[-\frac{(\frac{5}{3})^s}{C}\right]\equiv\exp\left[-\frac{r^\nu}{C}\right], \qquad \nu=\frac{\ln(\frac{5}{3})}{\ln 2}, \quad C\propto K\ . \tag{13}$$

Such a law of the decay of the correlation function signifies that there is no quasi-long-range order in the system. But since we are in the harmonic approximation this means that for the fractal network one can have no hope to have a better phase coherence than to have $g(r)$ of the form (13) with $\nu<1$. However there still remains a possibility to have a phase transition into a state with a more rapid decay of $g(r)$.

III. VORTICES

The unrestricted decrease of the coupling constant in the process of the renormalization suggests that for any temperature the harmonic approximation will not be applicable at large enough scales ($d\geq\xi\equiv K^{1/\nu}$). For XY model this usually means that one should also take vortices into consideration. It can be most easily done for the partition function (2) which allows exact decomposition into spin-wave part and vortex part:[8]

$$Z_{XY}=Z_{\rm SW}Z_{\rm vort} \tag{14}$$

with Z_{SW} coinciding in the form with the partition function of the harmonic approximation and

$$Z_{\rm vort}=\sum_{\{n_u\}}\exp\left[-\tfrac{1}{2}\sum_{u,u'}n_u\tilde{G}_{uu'}n_{u'}\right], \tag{15}$$

where integer variables n_u stand for the topological charges of the vortices and are defined on the plaquettes $\mathbf{u}$. Now $\tilde{G}_{uu'}$ is proportional to the lattice Green function of the dual lattice.

As in the case of the original lattice, the calculation of $\tilde{G}$ can be made with the help of the generating function, which is quite analogous to Eq. (8):

$$\tilde{Z}\{f\}=\prod_u\left[\int_{-\infty}^{\infty}dx_u\right]\exp\left[\frac{1}{2K}\sum_{\langle u,u'\rangle}(x_u-x_{u'})^2+2\pi i\sum_u f_u x_u\right], \tag{16}$$

where the first summation is now over pairs of nearest neighbors on the dual lattice. Then $\tilde{G}_{uu'}$ can be expressed as

$$\tilde{G}_{uu'}=-\left[\frac{\partial}{\partial f_u}\frac{\partial}{\partial f_{u'}}\tilde{Z}\{f\}\right]_{f=0}. \tag{17}$$

The first step is to integrate out the variables defined at the elementary plaquettes. That yields

$$\tilde{Z}\{f\}\propto\tilde{V}_1\{f\}\prod_u\left[\int_{-\infty}^{\infty}dx_u\right]\exp\left[-\frac{1}{5K_1}\sum_{\langle\langle u,u'\rangle\rangle}(x_u-x_{u'})^2+2\pi i\sum_u f'_u x_u\right] \tag{18}$$

with

$$K_1=\tfrac{3}{5}K, \quad f'_u=\tfrac{1}{3}\sum_{u'}f_{u'}+f_u\ , \tag{19}$$

where the sum over $\mathbf{u}'$ is to be taken over all elementary plaquettes surrounding the sth order plaquette $\mathbf{u}$. The form of the expression for f' corresponds to splitting each field f_u defined on an elementary plaquette into three equal parts that shift to the three neighboring plaquettes.

Thus we have obtained again the lattice that is dual to a Sierpinski gasket of a smaller order. But now we have coupling not only between nearest neighbors but also between second-nearest neighbors (see Fig. 3) which correspond to plaquettes of different size which are adjacent to the same elementary plaquette. All the couplings are equal. The exponent of the factor

$$\tilde{V}_1\{f\}=\exp\left[\sum\frac{4\pi^2K}{2}\frac{1}{3}(f_u)^2\right] \tag{20}$$

gives the first additive contribution to $\tilde{G}_{uu'}$, where the summation is to be taken over all elementary plaquettes.

Luckily at all other steps of the integration the struc-

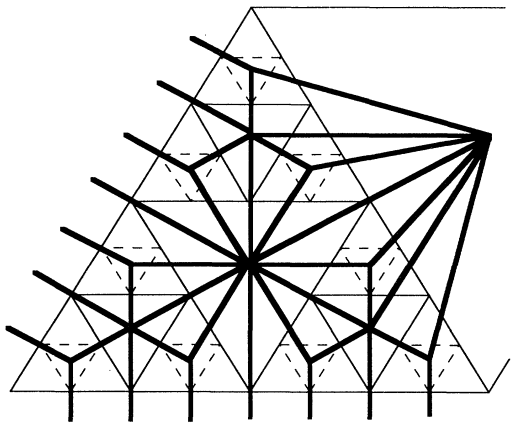

FIG. 3. Pairs of sites of the dual lattice that become coupled after the first decimation are connected by bold lines.

ture of the expression for $\tilde{Z}_s\{f\}$ will be conserved. No additional coupling will appear, and K will scale with a factor of $\frac{3}{5}$. So the scaling on the dual lattice and on the original lattice are completely consistent with each other. After summing all contributions $\tilde{G}$ can be expressed as follows:

$$\tilde{G}_{\mathbf{u}\mathbf{u}'}=4\pi^2K\left[\tfrac{1}{3}\delta_{\mathbf{u}\mathbf{u}'}+\tfrac{1}{2}\sum_{s=s_{\mathbf{u}\mathbf{u}'}}^{\infty}(\tfrac{3}{5})^{s-1}a_{\mathbf{u}}(s)a_{\mathbf{u}'}(s)\right], \tag{21}$$

where $s_{\mathbf{u}\mathbf{u}'}$ is the minimal order of a piece of fractal including both $\mathbf{u}$ and $\mathbf{u}'$. If $s=s_{\mathbf{u}\mathbf{u}'}$ then $a_{\mathbf{u}}(s)=1$ and if $s>s_{\mathbf{u}\mathbf{u}'}$ then $a_{\mathbf{u}}(s)$ is the contribution, after s transformations, of $h_{\mathbf{u}}$ to the sth order plaquette belonging to the sth order fractal containing $\mathbf{u}$. That means that $a_{\mathbf{u}}(s)$ has the form

$$a_u(s)=\sum_l(\frac{1}{3})^{N_l}, \tag{22}$$

where summation is to be taken over all paths that begin at $\mathbf{u}$, go step by step to a larger plaquette along the second-nearest neighbors (except for the first step in the case where $\mathbf{u}$ is an elementary plaquette when the paths go to a nearest neighbor) and reach the sth order plaquette which is at the center of the s-order fractal including $\mathbf{u}$. N_l is the number of steps forming the lth path. Then $\tilde{G}_{\mathbf{u}\mathbf{u}'}$ is always finite because it is evident that $a_{\mathbf{u}}(s)$ never exceeds 1. Therefore

$$\frac{1}{4\pi^2K}\tilde{G}_{\mathbf{u}\mathbf{u}'}<\tfrac{1}{3}+\tfrac{1}{2}\sum_{s=0}^{\infty}(\tfrac{3}{5})^s=\tfrac{19}{12} \tag{23}$$

for any $\mathbf{u}$ and $\mathbf{u}'$.

The same approach as in Sec. II can be used to understand the dependence of the vortex self-energy ($\propto\tilde{G}_{\mathbf{u}\mathbf{u}}$) on the plaquette size and that of the interaction energy ($\propto\tilde{G}_{\mathbf{u}\mathbf{u}'}$) on the separation distance $|\mathbf{u}-\mathbf{u}'|$. One considers two plaquettes $\mathbf{u}$ and $\mathbf{u}'$ with the order s' and s'', respectively. Then one performs the renormalization procedure $s=s_{\mathbf{u}\mathbf{u}'}$ times, obtaining the expression of Z_{vort} for the model with $K_R=(\frac{3}{5})^SK$ and distance between plaquette centers equal to $(\frac{1}{2})^S|\mathbf{u}-\mathbf{u}'|$. This shows that the interaction energy decays as

$$\tilde{G}(r)\propto(\tfrac{3}{5})^S\equiv r^{-\nu},\quad r>0, \tag{24}$$

with the same exponent $\nu=\ln(\frac{5}{3})/\ln 2$ as the correlation function of the harmonic approximation. This also means that the vortex self-energy is multiplied by a factor $\frac{3}{5}$ when the plaquette order is increased by one.

Our calculation has shown that the energy of a vortex is always finite on a Sierpinski gasket. Thus free vortices will always be present and no phase transition related to the dissociation of vortex molecules will take place (in contrast to the case of regular 2D lattice). Moreover the energy of a vortex scales with a factor $\frac{3}{5}$ with the increase of the size of the plaquette. So for any temperature for large enough plaquettes the concentration of vortices will be large in comparison with unity. So, at least starting from the scale ξ, the behavior of the correlation function will be strongly modified by vortices. In order to investigate the decay of correlation function at large scales we shall introduce another approximation, which is suitable for the limit of weak coupling.

IV. EXACTLY RENORMALIZABLE MODEL

Let us consider an XY model on a Sierpinski gasket described by the Hamiltonian

$$H=-k_BT\sum\ln\left[1+K\sum\cos(\theta_{\mathbf{r}}-\theta_{\mathbf{r}'})\right], \tag{25}$$

where the external sum is to be taken over all elementary plaquettes and the internal one over the perimeter of each such plaquette. If one develops the logarithm in powers of K, the first-order terms would give the Hamiltonian (1), while the higher-order terms will introduce also three-particle interaction on each elementary plaquette. Thus one can consider the Hamiltonian (25) as an approximation to the Hamiltonian (1) (which is exact in the limit $K\to 0$). The statistical weights $W=\exp(-H/k_BT)$ defined by (25) remain positive only for $K<K_*=\frac{2}{3}$, so it cannot be used directly for the investigation of the low-temperature properties of (1).

Hamiltonian (25) allows for the same exact renormalization as the Hamiltonian (4) of the harmonic approximation. But in contrast to the harmonic case the vortices are now also implicitly taken into account, because the phase-phase coupling in (25) is really periodic. In the case of (25) after performing the integration over first-order variables one obtains the Hamiltonian with the same structure, but with the rescaled value of coupling:

$$K_1=\frac{2+K}{4+K^3}K^2. \tag{26}$$

For $K\ll 1$ Eq. (26) reduces to $K_1\approx(\frac{1}{2})K^2$, so if one starts from $K_0\ll 1$ after s steps one obtains

$$K_s\approx 2(K_0/2)^{2^s}.$$

From Eq. (26) follows that for any $K<K_*$ the renormalized coupling is smaller than the initial one. This

means that the coupling always scales down to zero and there is no phase transition.

As for the calculation of the correlation function, the simplest thing to do would be to estimate its value for the sites on the corners of the same sth order plaquette (that is, at distance $r=2^s$). Then after performing the renormalization s times one will have them as the nearest neighbors, so the correlation function will be proportional to renormalized coupling:

$$g(r) \propto K_s \propto \exp\left[-\left[\ln\frac{2}{K_0}\right] r\right] . \tag{27}$$

That means that the correlations decay in an ordinary exponential way and that correlation radius R_c is equal to $1/\ln(2/K_0)$.

The decay of the correlations, described by Eq. (27) turned out to be more rapid than in the domain of the validity of the harmonic approximation. That can be considered as the manifestation of the finite concentration of vortices. Equation (27) will hold for the model (25) with $K \ll 1$, that is for the model (1) with $J \ll k_B T$. For $K \sim K_*$ it will give $R_c \sim 1$.

If one is interested in the low-temperature properties of models (1) or (3) one can use the harmonic approximation renormalization till K_R becomes of the order of one and then switch to the model (25) and use the fact that for it R_c will be of the order of one in the units of $\xi = K^{1/\nu}$. The correlation function will decay according to Eq. (13) for $r \ll \xi$ and according to Eq. (27) with $R_c \sim K^{1/\nu}$ for $r \gg \xi$.

V. CONCLUSION

We have studied an XY model on a Sierpinski gasket. A main difference with the case of a regular lattice is that the self-energy of a vortex is finite and the interaction energy decays as a power law of the distance. Therefore no phase transition can take place.

For any temperature the correlation function decays exponentially at large distances [$\exp(-r/R_c)$]. But in case of small temperature for distances smaller than the correlation radius ($R_c \sim (J/k_B T)^{1/\nu}$) it decays as $\exp(-r^\nu/C)$ with $C \propto K$ and $\nu = \ln(\frac{5}{3})/\ln 2 < 1$.

On the other hand in the case of a regular lattice formed by the sth order fractals (as in Ref. 5) the Berezinskii-Kosterlitz-Thouless transition should take place, but at a quite different temperature than in the periodic triangular lattice with the same lattice parameter. One should bear in mind that such complex lattice would be equivalent to the triangular lattice with the coupling constant rescaled by the factor of $(\frac{3}{5})^s$.

The approach developed in this paper can be expected to be of relevance for the description of superconducting wire networks only if the coherence length is not small in comparison with the wire width. In the opposite case the network will behave like a bulk sample without manifestation of fluctuations.

ACKNOWLEDGMENTS

This work has been supported by the Swiss National Science Foundation.

[1]For a recent review, see Coherence in Superconducting Networks, NATO advanced research workshop, Delft 1987 [Physica B **152** (1988)].

[2]J. M. Gordon, A. M. Goldman, J. Maps, D. Costello, R. Tiberio, and B. Whitehead, Phys. Rev. Lett. **56**, 2280 (1986).

[3]P. Martinoli *et al.* (private communication).

[4]S. Alexander, Phys. Rev. B **27**, 1541 (1983).

[5]S. Alexander and E. Halevi, J. Phys. (Paris) **44**, 805 (1983).

[6]V. L. Berezinskii, thesis, L. D. Landau Institute, Moscow, 1971.

[7]J. Villain, J. Phys. (Paris) **36**, 581 (1975).

[8]B. Nienhuis, J. Stat. Phys. **34**, 731 (1984).