Magnetoinductance of a superconducting Sierpinski gasket

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A study of the magnetoinductance $L(B)$ of a planar superconducting fractal lattice, the Sierpinski gasket (SG), exposed to a perpendicular magnetic field B is reported. Being inversely proportional to the superfluid density in the gasket, $L(B)$ provides a tool to appreciate how frustration effects created by B and characterized by a parameter $f \propto B$ affect phase coherence in a superconductor sharing essential geometrical elements with a truly percolating system near threshold. Both Josephson junction arrays (JJA) and superconducting wire networks (SWN) differing in their current-phase relations are considered and described in terms of interacting phase variables associated with the sites of the gasket. Relying on a mean-field approach, two central issues are addressed: the fine structure of $L(f)$ reflecting flux-quantization phenomena in loops with a hierarchical distribution of sizes and the low-field ($f \rightarrow 0$) scaling behavior of $L(f)$ resulting from the self-similar geometry of the gasket. It is shown that for a particular set of f values consistent with the requirement of fluxoid quantization in the central loop of a gasket generated by repeated juxtapositions of gaskets of lower order ($f = P/2.4^N$, where N is the gasket order and P an integer) the problem of computing $L(f)$ reduces to a calculation on a finite gasket and can be solved exactly once its ground-state phase configuration is known. Considerable simplification is achieved by making use of the triangle-star transformation of electric networks. The amplitude of the fine structure is found to depend crucially on the degree of anharmonicity of the phase interaction function. It vanishes (thereby implying that L is independent of f) in weakly coupled SWN with a strictly harmonic interaction and reaches its maximum strength in JJA with a cosinusoidal interaction. Using a perturbative decimation procedure which takes advantage of the self-similar structure of the SG, the frustration-induced inductance correction $\delta L(f)$ is predicted to scale as f^{ν} with $\nu = \ln(125/33)/\ln 4 \approx 0.96$ in the asymptotic limit ($f \rightarrow 0$). This exact result as well as other theoretical predictions emerging from the model are found to agree with high-resolution measurements of $L(f)$ performed on triangular arrays of periodically repeated gaskets of proximity-effect coupled Pb/Cu/Pb Josephson junctions.

I. INTRODUCTION

The concept of fractal structure provides a very useful geometrical tool to describe some of the features of random systems.[1] For instance, percolating materials exhibit, near the percolation threshold, a natural self-similar structure with geometrical inhomogeneities occurring over a broad range of length scales. They can therefore be described by a family of scale-invariant lattices, such as the Sierpinski gasket originally proposed by Gefen *et al.*[2] to mimic the topological properties of the percolating cluster's backbone.

With regard to superconductivity, fractal concepts have proven to be instructive in getting some insight into the physics of granular superconductors near percolation.[3-6] These materials, usually conceived as arrays of randomly distributed superconducting grains weakly coupled by the Josephson effect,[7] exhibit intriguing magnetic properties arising from the combined effect of disorder and frustration.[8] Unfortunately, in most cases the structural aspects of randomness in real superconductors are poorly known making a detailed comparison of theory and experiment almost impossible. With the advent of modern microfabrication techniques, however, it has become possible to investigate model systems, such as Josephson junction arrays (JJA) and superconducting wire networks (SWN), where both the nature and the amount of disorder can be accurately controlled and the level of frustration continuously tuned via an external magnetic field B.

Within the vast family of systems with fractal features, the Sierpinski gasket (SG), because of its simple hierarchical structure deprived of the complexity resulting from randomness and its dilational symmetry, appears to be an excellent candidate to explore novel behavior emerging from fundamental ideas in statistical mechanics and condensed-matter physics. Early work on SG wire networks has focused on their mean-field superconducting-to-normal phase boundary $T_c(B)$ (Refs. 9 and 10) which was found to agree with calculations based on the Ginzburg-Landau theory.[10,11]

More recently, it has been shown[12,13] that the properties of vortices in fractals are fundamentally different from those of vortices in Euclidean systems. It turns out, in fact, that the energy required to create a vortex in a (triangular) loop of a given species h (h is the hierarchical

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index labeling a family of identical loops) scales with the size r_h of the loop as $r_h^{-\zeta}$ with $\zeta=\ln(\frac{5}{3})/\ln 2$, in striking contrast with the logarithmic divergence of the vortex nucleation energy in a genuine two-dimensional (2D) system. Impedance measurements[13] probing the dynamics of vortices in weakly frustrated SG networks over a range of length scales covering several levels of hierarchy in the gaskets were found to be consistent with the unusual scaling of the vortex energy in fractal lattices predicted by theory. This observation was confirmed in further work[14] where the dynamic response of an unfrustrated ($B=0$) SG in the critical region close to the transition was interpreted in terms of thermally nucleated vortices moving like Brownian particles in the hierarchical potential-energy landscape provided by the gasket.

In this paper we investigate, both theoretically and experimentally, the magnetoinductance $L(B)$ of a superconducting SG exposed to a weak perpendicular field B. The interest of this quantity resides in the observation that, being inversely proportional to the (size-dependent) superfluid density in the gasket, it provides a tool to appreciate how the degree of superconducting phase coherence in the system changes with B. $L(B)$ is therefore of considerable importance to understand the magnetic properties of a superconductor sharing essential geometrical elements with a truly percolating system. From previous experimental work[13] it is known that $L(B)$ exhibits a complex fine structure reflecting flux quantization phenomena in loops with a hierarchical distribution of sizes. Even more significant is the observation that, at very low fields, $L(B)$ exhibits scaling properties intimately related to the fractal structure of the gasket. Asymptotic scaling ($B\rightarrow 0$) and fine structure of $L(B)$ are the central issues we shall address in this paper. For simplicity, it will be assumed that screening currents are weak, so that B penetrates the gasket homogeneously. In this regime the system is uniformly frustrated, the degree of frustration being measured by a parameter f proportional to B.

The model we adopt throughout this paper relies on the usual description of JJA and SWN in terms of interacting phase variables associated with the sites of the gasket.[15] An exhaustive calculation of $L(f)$ should incorporate renormalization effects due to thermal fluctuations which, on account of the reduced (nontrivial) dimensionality of the SG, are expected to have a profound effect on phase coherence. While vortex fluctuations were studied in detail in connection with the critical behavior of the unfrustrated gasket ($f=0$), where they were found to suppress the Berezinskii-Kosterlitz-Thouless (BKT) transition,[12] the inclusion of fluctuation-induced renormalization phenomena at nonvanishing arbitrary frustrations appears to be a prohibitive task and will be ignored in this paper. Within this mean-field approach, the magnetoinductance merely depends on the structure of the ground state of the system (more precisely, on the ground-state distribution of the gauge-invariant phase differences across the links of the gasket). In this connection, it turns out that the requirement of fluxoid quantization in the central loop of a gasket resulting from the juxtaposition of three gaskets of lower order selects a particular set of frustrations for which the ground state of an infinite gasket can be constructed by replicating that of a finite gasket. Hence, for these f values the problem of calculating $L(f)$ reduces to a calculation on a finite gasket and can be solved exactly once its ground state is known. For computational purposes, it is useful to rely on an algorithm, the triangle-star transformations,[16] well known in the theory of electric networks.

Although particularly efficient for numerical calculations, this "direct method" does not provide any insight into the characteristic low-field scaling properties of the magnetoinductance resulting from the fractal geometry of the gasket. To study this interesting asymptotic limit ($f\rightarrow 0$), we take advantage of the self-similar structure of the system to develop a perturbative decimation method which reduces the problem of calculating the inductance corrections $\delta L(f)$ at low frustrations for a given gasket to an identical problem at higher frustrations for a gasket of lower order whose ground state can be easily determined. A major exact result emerging from this treatment is that in the asymptotic limit $\delta L(f)$ is predicted to scale with frustration as f^{ν} with $\nu=\ln(\frac{125}{33})/\ln 4\approx 0.96$. The fraction $\frac{33}{125}$ in the numerator of the exponent ν reflects the scaling of δL at each decimation step and is shown to arise from the first nonlinear (cubic) term in the current-phase relation of the individual junctions of the gasket (the argument of the logarithm in the denominator expresses the trivial renormalization of f resulting from the quadruplication of the cell areas at each decimation step).

To test the theoretical predictions, we have performed high-resolution magnetoinductance measurements on a periodic (triangular) array of fourth-order gaskets consisting of proximity-effect coupled Josephson junctions. At low temperatures, the asymptotic scaling of the data with frustration as well as the amplitude and the shape of the fine structure follow very closely the behavior predicted by theory. At higher temperatures, thermal fluctuations strongly enhance the fine structure, but have almost no influence on the mean-field scaling properties of $L(f)$ at low frustrations.

The paper is organized as follows. In Sec. II, we show that JJA and SWN behave as networks of inductances whose values are inversely proportional to the first derivative of the current-phase relation on the corresponding bond. In Sec. III, we describe the "direct method" to compute $L(f)$ using the triangle-star transformation. In Sec. IV, we introduce the concept of decimation and apply it to find the rules governing the scaling of the current-phase relation. In Sec. V, the decimation procedure is generalized to the case of a frustrated gasket and then used in a perturbative approach to find the asymptotic behavior of the inductance correction $\delta L(f)$. Magnetoinductance data are presented and discussed in Sec. VI.

II. INDUCTANCE OF NETWORKS AND ARRAYS

Quite generally, the inductance L of a conductor can be defined as a measure of the energy E required to drive a current I into the system:

$$E=\frac{L}{2}I^2 . \quad (1)$$

This definition is still valid for nonlinear systems provided one considers only small currents. For a network of inductive elements $\{L_{jj'}\}$, Eq. (1) can be generalized to

$$E=\sum_{jj'}\frac{L_{jj'}}{2}I_{jj'}^2, \quad (2)$$

where the index j labels the nodes of the network and the sum runs over its links $\{jj'\}$. Current conservation at each node j requires

$$\sum_{j'}I_{jj'}=0 \quad (I_{jj'}\equiv -I_{j'j}) . \quad (3)$$

To calculate the total inductance of a network, one needs to know how the currents $\{I_{jj'}\}$, are distributed in the system for a given value of the total current I. This problem is completely equivalent to that of finding the current distribution in a resistor network. It can be solved by minimizing, for a given I, the total energy of the system (in a resistor network the corresponding quantity is the total energy loss) with the requirement of current conservation at each node. Thus, the calculation of the total inductance of an inductor network is entirely analogous to that of the total resistance of a resistor network.[3] For a SG of order N (Fig. 1) consisting of identical resistances R it has been shown [2,3] that the resistance R_N between the vertices of the gasket is given by

$$R_N=\left[\frac{5}{3}\right]^N\frac{2R}{3} , \quad (4)$$

$2R/3$ being the resistance of the zeroth-ordr gasket (simple triangle). Therefore, the inductance of a SG formed by identical inductances L is given by the same equation with R replaced by L.

Let us now consider a superconducting network (JJA or SWN) with arbitrary current-phase relation on each bond exposed to an applied perpendicular magnetic field B. Since at the low temperature of interest in our mean-field approach only the phase of the complex order parameter is a relevant degree of freedom, the energy of the system can be expressed by a Hamiltonian of the form

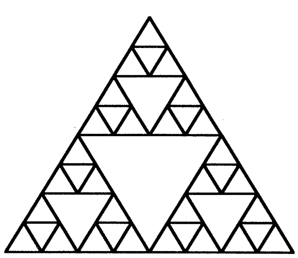

FIG. 1. Like other self-similar structures the Sierpinski gasket is defined by the recursive algorithm of its construction. A gasket of the order N is obtained by connecting three gaskets of order $(N-1)$ at their vertices. The construction starts from a simple triangle which plays the role of the zeroth-order gasket. The picture shows a third-order gasket.

$$H=\sum_{jj'}V_{jj'}(\varphi_j-\varphi_{j'}+A_{jj'}) , \quad (5)$$

where φ_j is the phase of the order parameter in the superconducting grain (or node) at the site j, $A_{jj'}$ is proportional to the line integral of the vector potential along the path connecting j to j', and $V_{jj'}(\theta)$ is an even periodic function of θ (with period 2π) depending on the form of the current-phase relation in the link jj'.

In the following, we shall neglect screening effects and assume that the $\{A_{jj'}\}$ are entirely determined by the vector potential of the external magnetic field B. Then, the sum of the $A_{jj'}$ along any closed path on the network is proportional to the area S enclosed by the path

$$\sum_{\square}A_{jj'}=\left[\frac{2\pi}{\phi_0}\right]BS , \quad (6)$$

where ϕ_0 is the superconducting flux quantum. If we introduce the notion of gauge-invariant phase difference $\theta_{jj'}$ across the bond jj',

$$\theta_{jj'}=\varphi_j-\varphi_{j'}+A_{jj'}\equiv-\theta_{j'j} , \quad (7)$$

condition (6) can be rewritten as

$$\sum_{\square}\theta_{jj'}=2\pi\left[\frac{S}{S_0}f-m\right], \quad (8)$$

where f is the magnetic flux (in units of ϕ_0) threading some reference cell of area S_0 and m is an integer accounting for the fact that the phase is defined only modulo 2π. Equation (8) is nothing but a manifestation of fluxoid quantization in a multiply connected superconductor. In the gauge-invariant description it is natural to reduce the $\{\theta_{jj'}\}$ to the interval $-\pi<\theta_{jj'}<\pi$. Then, in the ground state the integers $\{m\}$ can only be zero or positive (for positive f) and can be interpreted as the number of vortices penetrating a given cell.

In networks of thin superconducting wires the periodicity of $V(\theta)$ results from phase-slip processes which are bound to occur if θ tries to escape from the interval $-\pi<\theta<\pi$. This allows one to describe both JJA and SWN by Hamiltonians having the same structure and differing only in the functional dependence of the periodic interaction V on θ.

Variation of Eq. (5) with respect to φ up to second order in $\delta\varphi$ gives

$$\delta E=\frac{1}{2}\sum_{jj'}V''_{jj'}(\theta_{jj'})\delta\varphi_j-\delta\varphi_{j'})^2 , \quad (9)$$

where the contribution involving the linear terms vanishes because of current conservation at the nodes [Eq. (3)]. If we now relate $\delta\varphi_j$ to the variation $\delta I_{jj'}$ of the current $I_{jj'}=-(2e/\hbar)V'_{jj'}(\theta_{jj'})$ through the link jj',

$$\delta I_{jj'}=-(2e/\hbar)V''_{jj'}(\theta_{jj'})(\delta\varphi_j-\delta\varphi_{j'}) ,$$

we find that δE can be expressed as

$$\delta E=\frac{1}{2}\sum_{jj'}\left[\frac{\hbar}{2e}\right]^2\frac{1}{V''(\theta_{jj'})}(\delta I_{jj'})^2 . \quad (10)$$

Comparison of Eq. (10) with Eq. (2) shows that, as seen from an external current source, JJA and SWN behave as networks of inductances whose values are given by

$$L_{jj'}=\left[\frac{\hbar}{2e}\right]^2\frac{1}{V''_{jj'}(\theta_{jj'})} . \tag{11}$$

For a JJA the interaction functions $V(\theta)$ is determined by the energy-phase relation of a single junction and is of the form

$$V(\theta)=J(1-\cos\theta) , \tag{12}$$

where J is the Josephson coupling energy, the current across the junction being $I=(2e/\hbar)J\sin\theta$. It follows that for a JJA exposed to a perpendicular magnetic field the effective inductance of the junction jj' is given by

$$L_{jj'}=\left[\frac{\hbar}{2e}\right]^2\frac{1}{J\cos(\theta_{jj'})} . \tag{13}$$

Thus, even if all the junctions are identical, their effective inductances in the frustrated system may differ substantially from each other on account of the nonuniform distribution of the $\{\theta_{jj'}\}$.

On the other hand, in SWN the energy of a link (a piece of superconducting wire) is almost harmonic in θ, the anharmonic corrections being of the order of $(\xi/a)^2$, where ξ is the coherence length and a the length of the link.[17] Accordingly, in SWN the inductance modulation by a magnetic field should be much less pronounced than in JJA unless a becomes comparable to ξ. For simplicity, in the following we shall systematically omit the factor $(\hbar/2e)^2$ which should appear in all the explicit expressions of the inductance.

III. DIRECT CALCULATION OF THE INDUCTANCE OF SIERPINSKI GASKETS AND OF SIERPINSKI GASKET ARRAYS

In an ideal SG (Fig. 1) the area of the large cell located in the center of a gasket of order N is given by $4^{N-1}S_0$, where S_0 is the area of the smallest triangular cell. Since all the cell areas are multiples of S_0, it is convenient to use the magnetic flux f threading the smallest cell as a measure of the frustration of the system. From the structure of the Hamiltonian (5), it is clear that H is invariant with respect to changes in the flux of each cell corresponding to an integer number of flux quanta and also to changes in the sign of the magnetic field. Therefore, it is sufficient to focus only on the interval $0\le f\le\frac{1}{2}$.

For a particular set of frustrations, the structure of the ground state of the infinite gasket can be found by studying a finite gasket. We start with a gasket of order N at frustration f and assume that its ground state is symmetric (in terms of the gauge-invariant variables $\{\theta_{jj'}\}$) with respect to rotations of the system. Then, the sum Θ of the $\{\theta_{jj'}\}$ along each side of the gasket is given by

$$\Theta=\frac{2\pi}{3}(4^Nf-M) ,$$

where M is the sum of the variables $\{m\}$ in Eq. (8) and can be interpreted as the total number of vortices threading the gasket. Now, suppose one tries to construct the ground state of the gasket of order $(N+1)$ by juxtaposition of three Nth-order gaskets. One can then easily verify (see Fig. 2 for an illustration) that the constraint (8) imposed to the central cell of the new gasket is fulfilled only if $2\cdot4^Nf$ is an integer, that is only if

$$f=\frac{P}{2\cdot4^N} , \tag{14}$$

where P is an integer.

For a SWN with a linear current-phase relation it has been shown by Ceccatto *et al.*[3] that the ground state of a second-order gasket is always symmetric with respect to rotations of the system. However, even if the ground state of an Nth-order gasket (with $N>2$) were not symmetric at some frustration and, consequently, the variables $\{\Theta\}$ were different on the three sides of the gasket, the condition

$$\Theta_1+\Theta_2+\Theta_3=2\pi(4^Nf-M)$$

should still be satisfied. Moreover, for frustrations satisfying Eq. (14) the juxtaposition of three gaskets with nonsymmetric ground states (see Fig. 2) can be always performed in such a way that the constraint (8) will be also satisfied on the central loop of the resulting system. This means that for frustrations obeying Eq. (14) the condition (8) will be automatically fulfilled on the central loop of a gasket generated by repeated juxtapositions. This implies that the ground state of an infinite gasket will consist of "replicas" of the same state of a finite gasket. Thus, the structure of the ground state of an infinite gasket at frustrations given by Eq. (14) can be found by determining the ground state of a finite gasket. Having shown that the network inductance corresponding to any state is completely determined by the distribution of the $\{\theta_{jj'}\}$, we conclude that for frustrations satisfying Eq. (14) the problem of calculating the inductance reduces to a calcu-

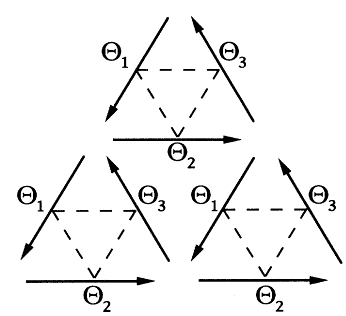

FIG. 2. In constructing a gaskets of order N by juxtaposition of three gaskets of order $(N-1)$, the arrows associated with the gauge-invariant phase differences on the central loop of the resulting system turn out to rotate in a direction opposite to that of the other loops. From the constraints imposed by fluxoid quantization [Eq. (8)] in the different loops it follows that the configuration shown in the figure is possible only if $2(\Theta_1+\Theta_2+\Theta_3)$ is a multiple of 2π. Obviously, this conclusion is still valid if the variables Θ_i $(i=1,2,3)$ are equal to each other.

lation on a finite gasket and can be solved exactly once the structure of its ground state is known.

As an illustration, let us calculate the inductance of a SG at $f=\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2}$. In this case it is sufficient to consider a first-order gasket formed by nine inductive elements whose inductances can take only two different values, L_1 and L_2 (Fig. 3). The current distribution in this simple network is easily determined so that, in terms of L_1 and L_2, its total inductance (for a current entering at one vertex and leaving at another one) can be written as

$$L=\frac{2L_1}{3}\frac{3L_1+2L_2}{2L_1+L_2} . \tag{15}$$

For a JJA the values of L_1 and L_2 at $f=\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2}$ follow from Eq. (13) by substituting the values of $\theta_{jj'}$ deduced from simple symmetry considerations. For instance, for $f=\frac{1}{4}$ one has $\theta_{jj'}=\pi/2$ on the three internal bonds and accordingly $L_2=\infty$, whereas $\theta_{jj'}=0$ on the six external bonds and therefore $L_1=1$ (in units of $1/J$). Then, using Eq. (15), one obtains

$$\frac{L(1/4)}{L(0)}=\frac{6}{5} .$$

The situation is even simpler for $f=\frac{1}{2}$, since $\theta_{jj'}=\pi/3$ on all the bonds and therefore

$$\frac{L(1/2)}{L(0)}=2 .$$

To eliminate the trivial dependence on the gasket size, inductances are conveniently normalized to $L(0)$, the value for the unfrustrated system.

For frustrations f expressed by an irreducible fraction with denominators 16 or 32 it is sufficient to consider a second-order gasket. In this case one has first to solve a system of equations (one of which is, in general, nonlinear) in order to find the structure of the ground state at the frustration of interest and then to calculate the inductance of a network of 27 inductors having five different values which is rather cumbersome. Considerable simplification of the second step of this program can be achieved by making use of the so-called triangle-star transformation which is well known in the theory of resistive networks.[16] An elementary step of this transformation is shown in Fig. 4(a). The triangle formed by the inductances L_1, L_2, and L_3 can be replaced by the star formed by the inductances

$$l_i=\frac{L_1L_2L_3}{(L_1+L_2+L_3)L_i} , \tag{16}$$

where $i=1,2,3$. The inverse transformation is given by

$$L_i=\frac{l_1l_2+l_2l_3+l_3l_1}{l_i} . \tag{17}$$

For an Nth-order gasket, $(N+1)$ successive applications of the triangle-star transformation (16) reduce the gasket [as shown in Fig. 4(b)] to a simple "star" whose total inductance (between the star vertices) can be trivially calculated, the inductive elements being now connected in series.

In Fig. 5, we present numerical calculations of the normalized ground-state energy (per elementary bond) and inverse inductance of a JJA with cosinusoidal interaction [Eq. (12)]. The calculations were performed on a third-order gasket at multiples of $f=\frac{1}{128}$ [Eq. (14)]. According to our previous discussion, these results are valid also for gaskets of larger order ($N>3$), including the infinite SG.

In the experiments discussed later on in this paper (Sec. VI) the *sheet inductance* of a regular triangular lattice of identical gaskets of order N' connected at the vertices was investigated. For frustrations satisfying Eq. (14) with $N\leq N'$ the structure of the ground state of such a system is also determined by the structure of the ground state of a single gasket of order N. Since the sheet inductance of a regular gasket array is proportional to the inductance of its constituent gaskets, the results of the calculation performed on the third-order gasket (Fig. 5) are also valid for the normalized sheet inductance of the corresponding composite periodic system.

Although the direct inductance calculation presented in this section can in principle be performed for a gasket of arbitrary (finite) size, the problem of finding the structure of the ground state becomes more and more complex with increasing gasket order. In fact, the number of equations to be solved to determine the structure of a

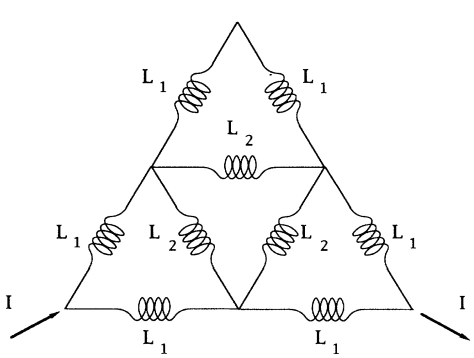

FIG. 3. To calculate the inductance of a Sierpinski gasket at frustrations corresponding to multiples of $f=\frac{1}{8}$, it is sufficient to consider a first-order gasket constructed with only two inductances L_1 and L_2.

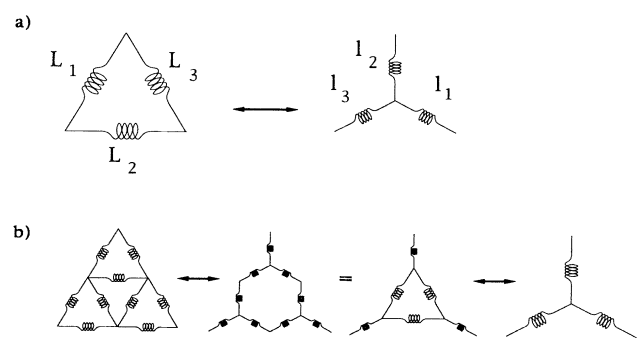

FIG. 4. (a) The triangle-star transformation and (b) its application to the calculation of the inductance of a first-order gasket.

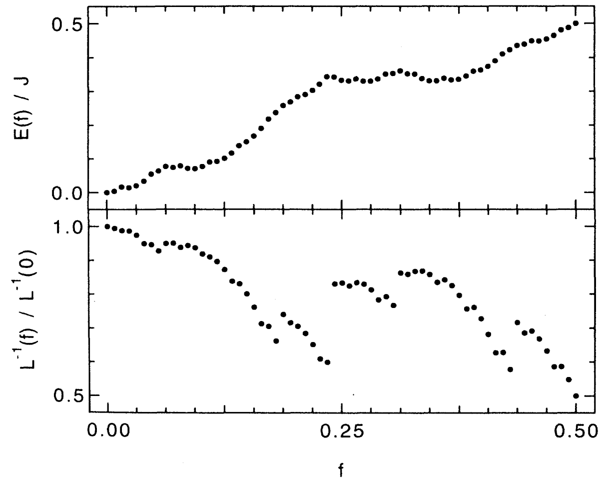

FIG. 5. Normalized ground-state energy (per bond) and inverse inductance of a Sierpinski gasket of Josephson junctions with a sinusoidal current-phase relation at frustrations corresponding to multiples of $f=\frac{1}{128}$.

symmetric ground state on an Nth-order gasket is equal to $(3^N+1)/2$, an expression showing very clearly why calculations become rapidly tedious with increasing N.

Even more significantly, numerical calculations prevent any insight into the asymptotic behavior of the inductance at small frustrations. To study the scaling properties of the inductance at small f, it is necessary to develop methods which take advantage of the self-similar nature of the SG. Such methods will be introduced in the next two sections.

IV. SCALING OF THE NONLINEAR CORRECTIONS TO THE CURRENT-PHASE RELATION

Like other fractals, the SG has a hierarchical self-similar structure. A second-order gasket can be thought of as a first-order gasket consisting of three first-order gaskets and so forth. This suggests that it should be possible to construct recursive relations allowing to express the properties of a given gasket in terms of those of a gasket of smaller order. In this section and in the following one we systematically develop such an approach and apply it to explore the asymptotic behavior of different quantities, including the frustration-dependent corrections of the inductance. For simplicity, in this section we start with the analysis of the unfrustrated system.

Let us consider a first-order SG. In the absence of a magnetic field it is described by the Hamiltonian

$$H=\sum_{\langle ij\rangle} V(\varphi_i-\varphi_j)\ , \qquad (18)$$

where the sum is taken over the nine pairs of nearest neighbors (Fig. 6). In the following we shall assume that, even if the gasket under consideration is part of a larger system (for instance, of a gasket of higher order or of a regular triangular lattice of first-order gaskets), it can be connected to the rest of the system only by its vertices (sites 1, 2, and 3 in Fig. 6). Therefore, since we are deal-

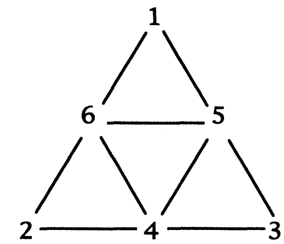

FIG. 6. Labeling of the nodes of the first-order gasket on which the decimation procedure is carried out. After decimation only the nodes 1, 2, and 3 survive.

ing with pairwise interactions, only the site variables φ_1, φ_2, and φ_3 can interact with the rest of the system whereas φ_4, φ_5, and φ_6 only interact with φ_1, φ_2, and φ_3. This means that, if we find the values of φ_4, φ_5, and φ_6 which minimize the energy of the first-order gasket for given values of φ_1, φ_2, and φ_3, we can then express its energy as a function of φ_1, φ_2, and φ_3 only. This procedure, which amounts to replacing the original first-order gasket by a zeroth-order gasket (an elementary triangle), is called decimation and can be applied to all the first-order gaskets composing a larger system. For example, decimation will transform an Nth-order gasket into an $(N-1)$th-order gasket.

For the case of a pure harmonic interaction

$$V(\theta)=\frac{J}{2}\theta^2 \qquad (19)$$

the minimization of the energy of the first-order gasket with respect to the "internal" variables φ_4, φ_5, and φ_6 can be performed exactly.[12] Variation of Eq. (18) with respect to φ_4, φ_5, and φ_6 results in a system of three linear equations whose solution is

$$\varphi_4=\frac{\varphi_1+2\varphi_2+2\varphi_3}{5}\ ,$$
$$\varphi_5=\frac{2\varphi_1+\varphi_2+2\varphi_3}{5}\ , \qquad (20)$$
$$\varphi_6=\frac{2\varphi_1+2\varphi_2+\varphi_3}{5}\ .$$

Then, substitution of Eq. (20) into the Hamiltonian leads to an expression of the form

$$H=V_R(\varphi_1-\varphi_2)+V_R(\varphi_2-\varphi_3)+V_R(\varphi_3-\varphi_1) \qquad (21)$$

showing that the energy is a sum of contributions associated with pairs of nearest neighbors on the decimated gasket. The renormalized interaction $V_R(\theta)$ differs from the original one [Eq. (19)] only in the coupling constant which is renormalized according to the rule[12]

$$J \Longrightarrow J_R=\tfrac{3}{5}J\ .$$

To illustrate how decimation works in a specific system, let us consider a triangular lattice of Nth-order gaskets as that studied in the experiments described in Sec. VI. By applying N times the decimation process, one is finally left with a regular triangular lattice with a renormalized coupling constant

$$J_R^{(N)}=(\tfrac{3}{5})^N J \ . \tag{22}$$

Since the inductance is inversely proportional to the coupling constant, Eq. (22) implies that the sheet inductance of the system scales with N as $(\frac{5}{3})^N$. This agrees with the conclusion reached in Sec. II [Eq. (4)].

In real systems the interaction function is not, in general, harmonic. Let us explore what happens if we add the next-order term to the harmonic approximation:

$$V(\theta)=\frac{J}{2}\theta^2-\frac{K}{24}\theta^4 \ . \tag{23}$$

Differentiation of $V(\theta)$ shows that K sets the amplitude of the first nonlinear correction to the current-phase relation:

$$I(\theta)\propto\frac{dV}{d\theta}=J\theta-\frac{K}{6}\theta^3 \ .$$

The equations for φ_4, φ_5, and φ_6 obtained by variation of the Hamiltonian (18) with the interaction (23) are nonlinear and can be solved only perturbatively, the first correction to the unperturbed solution [Eq. (20)] being of the form

$$\begin{aligned}\delta\varphi_4&=(K/J)F(\varphi_1-\varphi_3,\varphi_2-\varphi_1,\varphi_3-\varphi_2) \ ,\\ \delta\varphi_5&=(K/J)F(\varphi_2-\varphi_1,\varphi_3-\varphi_2,\varphi_1-\varphi_3) \ ,\\ \delta\varphi_6&=(K/J)F(\varphi_3-\varphi_2,\varphi_1-\varphi_3,\varphi_2-\varphi_1) \ ,\end{aligned} \tag{24}$$

where the function $F(\theta_1,\theta_2,\theta_3)\equiv F(\theta_2,\theta_1,\theta_3)$ is a polynomial of the third degree whose coefficients do not depend on the coupling constants J and K. Since the values of φ_i (with $i=4,5,6$) given by Eq. (20) are those minimizing the harmonic part of the Hamiltonian, substitution of $\varphi_i+\delta\varphi_i$ into this part of H only produces second-order corrections in K. First-order corrections in K only appear by substituting the solution into the fourth-order term of H and merely depend on the unperturbed solution (20). Thus, if we restrict our attention only to first-order terms in K, it is possible to derive the renormalized Hamiltonian simply by substituting Eq. (20) into H without knowing the explicit form of $F(\theta_1,\theta_2,\theta_3)$. It turns out that the renormalized Hamiltonian has the same form as the original one, but with the renormalized coupling constants:

$$J_R=\tfrac{3}{5}J, \quad K_R=\tfrac{99}{625}K \ . \tag{25}$$

Remarkably, the form of the fourth-order term remains the same, thereby leaving the pairwise nature of the interaction between the phases unaltered.

The perturbative treatment outlined above shows that a single decimation step of the SG reduces the relative amplitude $\gamma=K/J$ of the nonlinear term in the current-phase relation by a factor of $\frac{33}{125}$ which is significantly smaller than one. Therefore, even if the perturbative approach is not fully justified at the beginning of the decimation process, it becomes quantitatively correct after a few decimation steps.

Contributions to the Hamiltonian of higher order in θ are expected to decay with decimation even faster then the fourth-order term. For instance, it can be shown that, if two of the three variables φ_1, φ_2, and φ_3 are equal, the term of order $2n$ in $V(\theta)$ scales with decimation by a factor of

$$\lambda_{2n}=\frac{3^{2n}+2^{2n}+2}{5^{2n}} \tag{26}$$

which monotonically decreases with increasing n. Notice that for $n=1$ and $n=2$, Eq. (26) correctly predicts the scaling factors we found before [Eq. (25)] within the framework of the more general treatment.

Since $V(\theta)$ is periodic in θ, the fast decrease of the anharmonic corrections predicted by the perturbative treatment implies that, even if one starts from a cosinusoidal interaction [Eq. (12)], the shape of the effective interaction evolves with decimation towards a piecewise parabolic function which is significantly more "rigid" against large-amplitude thermal fluctuations than the cosinusoidal form. This means that in a triangular lattice consisting of periodically repeated gaskets of Josephson junctions fluctuations will be less relevant than in a triangular JJA with a coupling constant equal to the renormalized coupling constant of the gasket array [Eq. (22)].

As can be seen from Eq. (11), the nonlinear corrections to the current phase relation do not affect the inductance of an unfrustrated system. On the other hand, at nonvanishing frustrations currents are flowing in the system even in the absence of an external current. It is precisely the presence of nonharmonic terms in the Hamiltonian which provides the coupling between the externally driven and the internal currents. Since a single decimation step corresponds to a change in the effective frustration by a factor of 4, we expect that for $f\rightarrow 0$ the frustration-induced corrections to the inductance due to the presence of nonlinear terms in the current-phase relation should scale as f^{ν} with

$$\nu=\frac{\ln(125/33)}{\ln 4}\approx 0.96 \ , \tag{27}$$

where the fraction $\frac{33}{125}$ reflects the scaling of the inductance corrections one expects at every decimation step. In the next section we generalize the decimation procedure to the case of a frustrated gasket and show that the power-law asymptotic behavior of the inductance corrections corresponds indeed to the exponent given by Eq. (27).

V. DECIMATION OF THE GASKET IN FRUSTRATED SYSTEMS AND ASYMPTOTIC BEHAVIOR OF THE INDUCTANCE

A. Transformation of the harmonic Hamiltonian

At nonvanishing frustrations the system can be described in terms of the gauge-invariant bond variables $\theta_{jj'}\equiv-\theta_{j'j}$ [Eq. (7)]. Once more, it is convenient to consider a first-order SG described by the Hamiltonian

$$H=\sum_{\langle ij\rangle} V(\theta_{jj'}) , \tag{28}$$

where the sum runs again over the nine pairs of nearest neighbors (Fig. 6). The $\{\theta_{jj'}\}$, however, should now obey constraints of the form (8):

$$\begin{aligned}&\theta_{16}+\theta_{65}+\theta_{51}=2\pi f, \quad \theta_{24}+\theta_{46}+\theta_{62}=2\pi f ,\\ &\theta_{35}+\theta_{54}+\theta_{43}=2\pi f, \quad \theta_{45}+\theta_{56}+\theta_{64}=2\pi f .\end{aligned} \tag{29}$$

In writing these expressions we have assumed that there are no vortices inside the cells of the gasket [i.e., $m=0$ in Eq. (8)], which is certainly correct in the limit of low frustrations in which we are interested here. After decimation, it is necessary to introduce a set of new $\{\theta_{jj'}\}$ defined on the bonds of the decimated (zeroth-order) gasket and related to the old variables by (see Fig. 6):

$$\begin{aligned}&\theta_{12}=\theta_{16}+\theta_{62} ,\\ &\theta_{23}=\theta_{24}+\theta_{43} ,\\ &\theta_{31}=\theta_{35}+\theta_{51} .\end{aligned} \tag{30}$$

The new θ variables should also satisfy a constraint of the form (8):

$$\theta_{12}+\theta_{23}+\theta_{31}=2\pi f_R ,$$

where we have introduced the "renormalized" frustration $f_R=4f$.

In the harmonic approximation (19) the minimization of the Hamiltonian (28) under the constraints (29) and (30) can be performed exactly and leads to

$$\begin{aligned}&\theta_{16}=\frac{5\theta_{12}+\theta_{23}-\theta_{31}}{10} ,\\ &\theta_{65}=\frac{-3\theta_{12}+\theta_{23}-3\theta_{31}}{20} ,\\ &\theta_{51}=\frac{-\theta_{12}+\theta_{23}+5\theta_{31}}{10}\end{aligned} \tag{31}$$

and to analogous expressions for the other six bond variables of the original (first-order) gasket which follow from Eq. (31) by cyclic permutations of the indices on the rhs. Then, substitution of the solution in the harmonic Hamiltonian gives

$$\sum_{\langle ij'\rangle}\frac{J}{2}\theta_{jj'}^2=\frac{J}{2}\left[\frac{3}{5}(\theta_{12}^2+\theta_{23}^2+\theta_{31}^2)-\frac{1}{5}(2\pi f)^2\right] . \tag{32}$$

This expression exhibits the same scaling factor ($\frac{3}{5}$) of the coupling constant which was obtained in the previous section for the unfrustrated system. It also shows that, if $V(\theta)$ is harmonic, the presence of frustration does not change the functional form of the renormalized Hamiltonian (for a given frustration the second term on the rhs of Eq. (32) simply causes an overall shift of the energy). Since the inductive properties of the system are determined by the second derivative of the energy [see Eq. (11)] and are therefore insensitive to a global energy shift, this result confirms our previous conclusion that the inductance of a gasket characterized by a purely harmonic interaction is independent of frustration.

B. Dependence of the ground-state energy on frustration

Decimation transforms a gasket of order N at frustration f into a gasket of order $(N-1)$ at frustration $f_R=4f$. For f low enough for no vortices to penetrate the first-order gaskets forming the Nth-order gasket, Eq. (32) allows one to relate the ground-state energies of the two systems without knowing the structure of their ground states. Moreover, since an infinite SG is self-similar, we can also use Eq. (32) to find a relation between the ground-state energies of an infinite gasket at frustration differing from each other by a factor of 4:

$$E(f)=\frac{1}{5}\left[E(4f)-\frac{J}{6}(2\pi f)^2\right] , \tag{33}$$

where the rhs has been multiplied by a factor of $\frac{1}{3}$ to account for the ratio between the number of bonds in the decimated gasket and in the original one.

Iteration of the recursive relation (33) leads to the following expression for the ground-state energy of a gasket characterized by a linear current-phase relation:

$$\begin{aligned}E(f/4^N)=&\left[E(f)-\frac{2\pi^2}{33}Jf^2\right]\left[\frac{1}{5}\right]^N\\ &+\frac{2\pi^2}{33}Jf^2\left[\frac{1}{16}\right]^N ,\end{aligned} \tag{34}$$

where $0<f<\frac{1}{2}$. The dependence on N in Eq. (34) shows that, for $f\to 0, E(f)$ obeys a power law, $E(f)\propto f^{\nu_E}$, with the exponent

$$\nu_E=\frac{\ln 5}{\ln 4}\approx 1.16 \tag{35}$$

and allows one to appreciate how fast the asymptotic regime is actually reached.

The value of the exponent ν_E was first calculated by Alexander and Halevi[5] by studying an expression for an upper bound to $E(f)$. In contrast to their work, Eq. (34) is an exact result. The structure of Eq. (34) implies that in a log-log plot $E(f)$ should look, in the limit $f\to 0$, as a tilted periodic function with a slope corresponding to the exponent (35). In the work of Meyer *et al.*[13] it was erroneously assumed that the same exponent describes the asymptotic scaling behavior of the frustration-dependent corrections to the inductance.

C. Transformation of the θ-dependent inductance corrections with gasket decimation

Let us now come back to a first-order gasket. At low frustrations (no vortices) the gauge-invariant phase differences $\{\theta_{jj'}\}$ are small on all the (nine) bonds. As a consequence, the corresponding inductances $\{L_{jj'}\}$ are almost equal and we can write

$$L_{jj'}=L_0+\delta L_{jj'}, \quad |\delta L_{jj'}|\ll L_0 .$$

To calculate the small inductance corrections $\{\delta L_{jj'}\}$ caused by frustration, we rely on a perturbative approach which takes advantage of the decimation scheme. A two-stage application of the triangle-star transformation [Fig. 4(b)] and, subsequently, of its inverse (star-triangle transformation) replaces the first-order gasket by a triangle (zeroth-order gasket) made up by the inductances:

$$\begin{aligned} L_{12} &= \tfrac{5}{3}L_0 + \tfrac{1}{9}[6(\delta L_{16}+\delta L_{62})+3(\delta L_{51}+\delta L_{24}) \\ &\quad -2(\delta L_{43}+\delta L_{35})+\delta L_{54}] , \\ L_{23} &= \tfrac{5}{3}L_0 + \tfrac{1}{9}[6(\delta L_{24}+\delta L_{43})+3(\delta L_{62}+\delta L_{35}) \\ &\quad -2(\delta L_{51}+\delta L_{16})+\delta L_{65}] , \\ L_{31} &= \tfrac{5}{3}L_0 + \tfrac{1}{9}[6(\delta L_{35}+\delta L_{51})+3(\delta L_{43}+\delta L_{16}) \\ &\quad -2(\delta L_{62}+\delta L_{24})+\delta L_{46}] , \end{aligned} \tag{36}$$

where only first-order terms in δL were retained and the two-index notation is the same as the one introduced in Sec. V A (see Fig. 6).

In the perturbative treatment it is sufficient to keep only the first nonharmonic term in the interaction function [as in Eq. (23)]. Then, by expanding Eq. (11) to first order in K, the inductance of each bond *before* decimation can be related to the corresponding gauge-invariant phase difference θ by

$$L(\theta)=\frac{1}{J}+\delta L(\theta), \quad \delta L(\theta)\approx \frac{K}{2J^2}\theta^2 . \tag{37}$$

Since we know how the $\{\theta_{jj'}\}$ transform in the decimation process [Eq. (31)], we can apply Eq. (36) to find the inductance *after* decimation without actually knowing how the $\{\theta_{jj'}\}$ are distributed in the ground state. In fact, by substituting Eq. (37) into Eq. (36) and using Eq. (31), it turns out that the inductance $L^{(1)}$ of each bond of the decimated gasket can be expressed as a function of the corresponding gauge-invariant phase difference θ as

$$L^{(1)}(\theta)=\frac{5}{3J}[1+a_1-b_1\theta+\frac{c_1}{2}\theta^2] , \tag{38}$$

where

$$\begin{aligned} a_1 &= \frac{161}{750}(2\pi f)^2\gamma , \\ b_1 &= \frac{67}{375}(2\pi f)\gamma , \\ c_1 &= \frac{33}{125}\gamma , \end{aligned} \tag{39}$$

and $\gamma=K/J$ is the ratio of the unrenormalized coupling constants in Eq. (23).

Let us now turn to larger gaskets, as we did in the previous subsection. Having demonstrated that the inductance of each bond of a decimated (zeroth-order) gasket only depends on the value of the corresponding θ, we can iterate the decimation procedure as long as the renormalized frustration is low enough for no vortices to penetrate the constituent first-order gaskets participating in the decimation process. If the decimation procedure is repeated N times, we get

$$L^{(N)}(\theta)=\left[\frac{5}{3}\right]^N \frac{1}{J}\left[1+a_N-b_N\theta+\frac{c_N}{2}\theta^2\right] , \tag{40}$$

where

$$\begin{aligned} a_N(f_R) &= [\tfrac{35\,461}{703\,638}(\tfrac{33}{125})^N+\tfrac{1139}{2619}(\tfrac{7}{100})^N-\tfrac{10\,561}{21\,762}(\tfrac{1}{16})^N](2\pi f_R)^2\gamma , \\ b_N(f_R) &= \tfrac{67}{291}[(\tfrac{33}{125})^N-(\tfrac{7}{100})^N](2\pi f_R)\gamma , \\ c_N(f_R) &= (\tfrac{33}{125})^N\gamma , \end{aligned} \tag{41}$$

and $f_R=4^Nf$ is the renormalized frustration. For $N=1$, Eq. (41) reduces to Eq. (39), as it should.

Notice that on the decimated gasket $L(\theta)$ is no longer an even function of θ. Since $L(\theta)$ is related to $V(\theta)$ by Eq. (11), it immediately follows that the renormalized interaction function $V_R(\theta)$ also loses its inversion symmetry. This is quite natural in a frustrated system where the vector character of the magnetic field enters the decimation procedure. As a consequence, the sign of the linear term in $L(\theta)$ is determined by the direction of the field.

D. Asymptotic behavior of the inductance at small frustrations

In the previous subsection we have shown that the problem of calculating the inductance corrections at low frustrations for a given gasket can be reduced to an identical problem at higher frustrations for a gasket of lower order where the distribution of the $\{\theta_{jj'}\}$ in the ground state may be already known. For instance, if we start from $f=1/2^{2N+1}$ and make N decimations, the effective frustration f_R of the decimated gasket will be $f_R=\frac{1}{2}$. At this frustration the ground state corresponds to $\theta=\pi/3$ on all the bonds of the gasket (see Sec. III), so that the inductance change $\delta L(f)\equiv L(f)-L(0)$ at $f=1/2^{2N+1}$ can be easily calculated by setting $\theta=\pi/3$ in Eq. (40):

$$\frac{\delta L(f)}{L(0)}=[\tfrac{3425}{117\,273}(\tfrac{33}{125})^N+\tfrac{1340}{2619}(\tfrac{7}{100})^N-\tfrac{10\,561}{21\,762}(\tfrac{1}{16})^N]\pi^2\gamma , \tag{42}$$

where $\gamma=1$ for a JJA with a cosinusoidal interaction [Eq. (12)], whereas for a SWN γ will be smaller (or even much smaller) than one. For $N=0$ ($f=\frac{1}{2}$) the numerical coefficient within the brackets on the rhs of Eq. (42) reduces to $\frac{1}{18}$ and for $N=1$ ($f=\frac{1}{8}$) to $\frac{19}{1440}$.

Quite similarly, if start from $f=1/2^{2N+2}$ and perform again N decimations, we end up with a gasket with an effective frustration $f_R=\frac{1}{4}$. At this frustration the ground state corresponds to $\theta=\pi/2$ on the three internal and to $\theta=0$ on the six external bonds of each constituent first-order gasket (see Sec. III). To calculate the relative inductance correction $\delta L(f)/L(0)$, we expand Eq. (15) for the inductance of a first-order gasket to first order in the corrections δL_i ($i=1,2$, see Fig. 3) and substitute Eqs. (40) and (41) obtaining, for $f=1/2^{2N+2}$,

$$\begin{aligned} \frac{\delta L(f)}{L(0)} &= \frac{14}{15}\frac{\delta L(\theta=0)}{L}+\frac{1}{15}\frac{\delta L(\theta=\pi/2)}{L} \\ &= a_N(\tfrac{1}{4})-b_N(\tfrac{1}{4})\frac{\pi}{30}+c_N(\tfrac{1}{4})\frac{\pi^2}{120} \\ &= [\tfrac{10\,024}{586\,365}(\tfrac{33}{125})^N+\tfrac{1474}{13\,095}(\tfrac{7}{100})^N-\tfrac{10\,561}{87\,048}(\tfrac{1}{16})^N]\pi^2\gamma . \end{aligned} \tag{43}$$

for $N=0$ ($f=\frac{1}{4}$) the numerical coefficient on the rhs of Eq. (43) reduces to $\frac{1}{120}$ and for $N=1$ ($f=\frac{1}{16}$) to $\frac{3463}{720\,000}$. Analogous calculations can be performed also for other families of frustrations whose values scale as 4^N with the result that the final expressions for the relative inductance variation have the same form as Eqs. (42) and (43), but with different coefficients in front of the powers of the scaling factors.

The asymptotic behavior of $\delta L(f)$ in the limit $f \to 0$ is determined by the first term in Eqs. (42) and (43). Thus, for $f \to 0$ the frustration-induced inductance correction obeys a power law, $\delta L(f) \propto f^{\nu}$, with the same exponent

$$\nu = \frac{\ln(125/33)}{\ln 4} \approx 0.96 \qquad (44)$$

we found earlier in Sec. IV [Eq. (27)]. We would like to stress that the leading terms in Eqs. (42) and (43) provide an exact asymptotic description (prefactors included) of $\delta L(f)/L(0)$ for the two specific frustration families considered above. It is for this reason that the numerical coefficients in these equations were expressed in their exact fractional form. It should also be noticed that our approach is not perturbative in the parameter γ, but in the inductance variations of the individual bonds of the original (undecimated) gasket which quickly fall off with decreasing frustration.

An alternative method to calculate $\delta L(f)$ within a perturbation scheme consists in substituting into the Hamiltonian $\{\theta_{jj'}\}$ of the form

$$\theta_{jj'} = \theta_{jj'}(f) + I\theta^{I}_{jj'} , \qquad (45)$$

where $\{\theta_{jj'}(f)\}$ and $\{\theta^{I}_{jj'}\}$ are, respectively, the distribution of the gauge-invariant phase differences in the ground state at a given frustration f and the distribution of the additional "phase twists" induced by an external current I flowing through the system. If Eq. (45) is substituted into the harmonic part of the Hamiltonian, mixed terms will never show up in the energy of the system as this quantity cannot be linear in I. As a consequence, the current contribution to the energy resulting from the harmonic part of the Hamiltonian is independent of frustration. However, when Eq. (45) is substituted into higher-order (nonharmonic) terms of the Hamiltonian, the internal currents of the frustrated gasket couple to the external current I, thereby providing a method to calculate $\delta L(f)$. After computing the $\{\theta_{jj'}(f)\}$ and the $\{\theta^{I}_{jj'}\}$ in the harmonic approximation, we have explicitly performed such a calculation for the fourth-order term in the Hamiltonian at some frustrations ($f=\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32}$) finding expressions for $\delta L(f)/L(0)$ identical to those given by Eqs. (42) and (43). Thus, the two perturbative methods developed in this work are essentially equivalent. However, the approach based on the coupling between internal and external currents shows more clearly why the asymptotic behavior of $\delta L(f)$ at small frustrations is entirely determined by the fourth-order term in the Hamiltonian. On the other hand, the decimation procedure is more appropriate to calculate $\delta L(f)$ at arbitrarily small frustrations.

Our results in the asymptotic regime imply that on a log-log plot $\delta L(f)/L(0)$ looks, at small f, like a tilted periodic function with a slope corresponding to the exponent ν given by Eq. (44). This is valid for both JJA and SWN, the only difference consisting in an overall downward shift of the curve for a SWN reflecting its lower value of γ. In order to estimate the amplitude of the oscillations, it is convenient to combine the decimation procedure with numerical calculations. For frustrations satisfying $f=P/2^{2N+1}$ with $5 \le P \le 16$ and $N>2$ we first proceed to $(N-2)$ decimation steps to reduce the problem to a second-order gasket whose inductance is then calculated with the direct method developed in Sec. III using Eqs. (40) and (41) to define the inductances of the bonds. The result, expressed in terms of the quantity $\Delta L^{-1} \equiv [L^{-1}(0) - L^{-1}(f)]$ measuring the change in superconducting phase coherence caused by frustration, is shown (open circles) in log-log form in Fig. 7 where it is compared with that obtained from a direct numerical calculation, based on Eq. (13), performed on a third-order gasket (solid circles). In order to appreciate how fast the asymptotic behavior is reached, we have subtracted the linear background by plotting, instead of $\log_{10}[\Delta L^{-1}/L^{-1}(0)]$, $\log_{10}[\Delta L^{-1}/L^{-1}(0)] - \nu \log_{10} f$ vs $\log_4 f$ [notice that $\Delta L^{-1}/L^{-1}(0) = \delta L/L(0)$ in the asymptotic limit].

It is seen that already in the second "decade" (or "hierarchical stage"), i.e., in the interval $\frac{1}{32} < f < \frac{1}{8}$, the behavior is very close to its asymptotic form. However, in the first "decade" ($\frac{1}{8} < f < \frac{1}{2}$) it is rather different from that observed in the other hierarchical stages. This is readily understood if one realizes that in this frustration range the $\{\theta_{jj'}\}$ on some bonds can be of the order of one. As a consequence, the exact value of the corresponding inductance, as given by Eq. (13), differs substantially from the approximation (37) which serves as a base for the

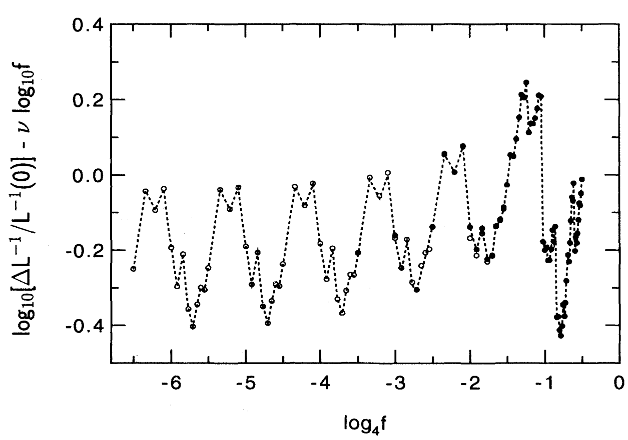

FIG. 7. log-log plot of the relative change of the inverse inductance of a Sierpinski gasket of Josephson junctions ($\gamma=1$) as a function of frustration (the linear background $\nu \log_{10} f$ has been subtracted). Solid circles: result of a direct numerical calculation performed on a third-order gasket. Open circles: result of a calculation based on the perturbative approach in which the problem is reduced to a calculation on a second-order gasket after performing one or more decimation steps. This result becomes asymptotically exact for $f \to 0$. The line connecting the circles is simply a guide to the eye.

asymptotic calculation. It should also be noticed that, already in the second hierarchical stage, the results of the asymptotic approach are very close to those emerging from the direct numerical calculation, thereby providing strong support for the reliability of the perturbation treatment developed in this section.

The peculiar behavior of the inductance in the first "decade" justifies our reluctance to comment on the self-similarity of $L(f)$ in Fig. 5. In fact, Fig. 7 clearly shows that, in order to make the self-similar nature of the results manifest, the numerical calculations should be performed at least on fourth- or fifth-order gaskets.

VI. COMPARISON WITH EXPERIMENT

In the previous sections we have developed various theoretical methods to calculate the mean-field inductance of a SG as a function of frustration. In particular, we have shown (see Fig. 7) that the hierarchical self-similar nature of the gasket becomes clearly manifest only in the asymptotic limit of very low frustrations where the inductance change is predicted to scale with f according to a well-defined power law reflecting the fractal geometry of the gasket. In this section we compare the theoretical predictions with high-resolution magnetoinductance measurements performed on triangular arrays of periodically repeated gaskets of proximity-effect coupled Josephson junctions. Under favorable experimental conditions, we have been able to resolve up to four hierarchical stages in the gasket magnetoinductance, thereby probing the inductive properties of the system in the low-frustration limit of interest.

A. Experimental details

The sample studied in this work is shown in Fig. 8. It consists of fourth-order gaskets sitting on the sites of a 78×78 triangular lattice and connected to each other at the vertices. Each gasket is an array of $3^4=81$ SNS junctions consisting of superconducting (S) Pb islands proximity-effect coupled to each other by an underlying

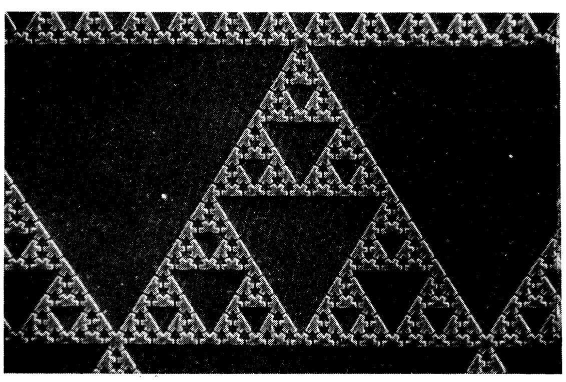

FIG. 8. Scanning electron micrograph showing a portion of a triangular array of periodically repeated fourth-order Sierpinski gaskets of proximity-effect coupled Pb/Cu/Pb Josephson junctions. The length of the elementary links of the gaskets is 8 μm. Notice that, with the exception of those centered on the common vertices of three constituent gaskets, the superconducting Pb islands have an asymmetric "truncated-star" shape.

Cu normal (N) layer. Length and width of the elementary links of the gaskets are, respectively, $a=8$ μm and $w=2$ μm. Since the ratio $D/\xi_n(T_{c0})$ between the junction gap $D(\approx0.72$ μm) and the normal-metal coherence length $\xi_n(T_{c0})$ (≈0.19 μm) at the transition temperature T_{c0} (≈7 K) of the individual superconducting islands is of the order of 4, the system is expected to behave as a genuine JJA with a sinusoidal current-phase relation over a relatively wide temperature range, $V(\theta)$ deviating significantly from Eq. (12) only at low temperature.[17] Because of its 2D nature at length scales larger than the gasket size, the unfrustrated array ($f=0$) exhibits a BKT transition at $T_c=5.93$ K.

Relying on a numerical inversion procedure, the sheet magnetoinductance of the gasket array was extracted from measurements of the mutual inductance change of a coaxially mounted drive-receive coil system due to the supercurrents flowing in the sample in response to a weak ac field.[18] The rms flux created by the driving ac current in an elementary triangular cell located just underneath the coils did not exceed $10^{-4}\phi_0$. This low-level excitation ensured a linear response and, combined with a suppression of ambient magnetic fields to ~1 mG, allowed f to be tuned with a precision better than 10^{-4}. Data were taken at 160 Hz with a sensitive SQUID-operated ac bridge. The major factor limiting the inductance resolution during our swept-frustration impedance measurements was the low-frequency noise generated by the solenoid providing the dc magnetic field. Typically, we have been able to resolve inductance changes of the order of 10 pH. The resolution was found to be somewhat better near the superconducting transition where, on account of the weaker screening effect provided by the sample, the gradiometer configuration of the receive coil suppressed external flux noise more efficiently than at lower temperatures.

The periodic arrangement of finite-order gaskets studied in this work ensured sample homogeneity over the macroscopic length scales set by the diameter (≈2 mm) of the detection coil. This is essential for the analysis of the diamagnetic response based on the inversion procedure to apply.[18] Moreover, the gasket order ($N=4$) is sufficiently high for the set of frustrations $f=P/2\times4^N=P/512$ [see eq. (14)] at which theory can be compared to experiment to be dense enough.

In the following, temperatures will be expressed in terms of the reduced temperature $\tau\equiv kT/J(T)$ relevant for the statistical mechanics of the system. The temperature-dependent Josephson coupling energy $J(T)$ was inferred from measurements of the "bare" sheet (kinetic) inductance $L(T)=(\hbar/2e)^2(5/3)^N/\sqrt{3}J(T)$ of the unfrustrated array at temperatures well below T_c. In terms of τ the BKT transition occurs at $\tau_c\approx0.23$ in good agreement with the theoretical prediction[12] $\tau_c=(3/5)^4\tau_{c0}$ based on Eq. (22), $\tau_{c0}\approx1.5$ being the reduced BKT transition temperature of a regular triangular lattice[19] with the same coupling energy $J(T)$.

B. Results and discussion

The inverse sheet inductance L^{-1} of our gasket array is shown in Fig. 9(a) as a function of the frustration pa-

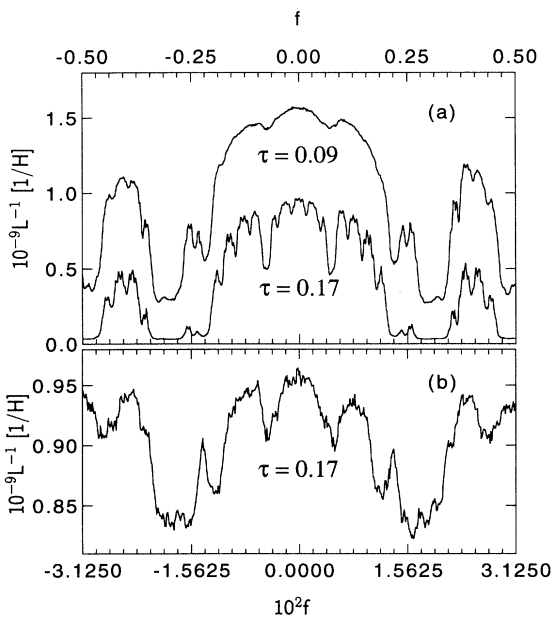

FIG. 9. (a) Inverse sheet inductance of the triangular lattice of fourth-order Sierpinski gaskets shown in Fig. 8 at two different reduced temperatures as a function of frustration. Both sets of data were taken at a driving frequency of 160 Hz. In (b) the flux axis has been expanded by a factor of 16 in order to highlight the scale-invariant behavior of the magnetoinductance resulting from the self-similar structure of the gaskets.

rameter f at two different reduced temperatures, $\tau=0.09$ ($T=5.51$ K) and $\tau=0.17$ ($T=5.81$ K). The data, periodic in f with period 1, exhibit a complex fine structure reflecting flux quantization in loops with a hierarchical distribution of sizes. This aspect is illustrated by comparing the inverse magnetoinductance data at $\tau=0.17$ of Fig. 9(a) with those shown in Fig. 9(b), where the flux axis was expanded by a factor of 16 to highlight the scale-invariant properties of the inductance resulting from the fractal geometry of the gaskets. The striking similarity of the two curves provides a clear demonstration of the self-similar structure of the gaskets, where the loop area S_h and, consequently, the magnetic flux scale as 4^{h-1} with the hierarchical index $h=1,2,\ldots,N$ labeling the different families of loops.

Although the overall shape of the inverse magnetoinductance curves at $\tau=0.09$ and 0.17 is quite similar [Fig. 9(a)], the fine structure is found to become richer and sharper with increasing temperature suggesting that thermal fluctuations play a major role in the description of superfluid and vortex dynamics. Similar behavior was observed also in wire networks of interconnected gaskets[13] and in regular triangular SNS arrays.[20] We interpret it as clear evidence that, at sufficiently high temperatures, phase coherence in the neighborhood of the "commensurate" ground states at $f=P/(2\times4^N)$, where the vortex lattice is pinned, is drastically disrupted by vortex-lattice defects, created by excess or missing vortices, moving almost freely on the pinned vortex background. This sharpens the fine structure substantially, thereby enhancing the amplitude of the oscillations.

In order to verify the asymptotic prediction of Sec. V, the logarithm of the relative inverse sheet inductance change $\Delta L^{-1}/L^{-1}(0)$, as deduced from the data of Fig. 9, is plotted against $\log_4 f$ in Fig. 10 and compared with the result of a calculation identical to that we described in connection with Fig. 7, where the direct method developed in Sec. III was combined with the perturbative decimation procedure based on Eqs. (40) and (41). At the lowest temperature ($\tau=0.09$), where thermal fluctuations are expected to be almost irrelevant, the general scaling of the data with frustration as well as the amplitude and the shape of the oscillations follow quite nicely the behavior predicted by the mean-field theory. However, on account of the limited inductance resolution $[\delta L/L(0)\approx1\%]$ attainable in the ac measurements, at $\tau=0.09$ we have been able to resolve only two hierarchical stages and to observe incipient asymptotic behavior merely in the second one ($\frac{1}{32}<f<\frac{1}{8}$). In this respect, the experimental conditions are much more favorable at higher temperatures, where thermal fluctuations not only enhance the amplitude of the oscillations, but also promote a richer fine structure, as demonstrated by the larger number (four) of self-similar stages emerging from the data at $\tau=0.17$ in Fig. 10 (notice that $h=4$ is actually the largest number of stages compatible with the order of our gaskets). A shown in Ref. 13, the evolution of the fine-structure richness with temperature finds a natural explanation in terms of thermal activation of the vortices in the potential-energy landscape created by the gaskets.

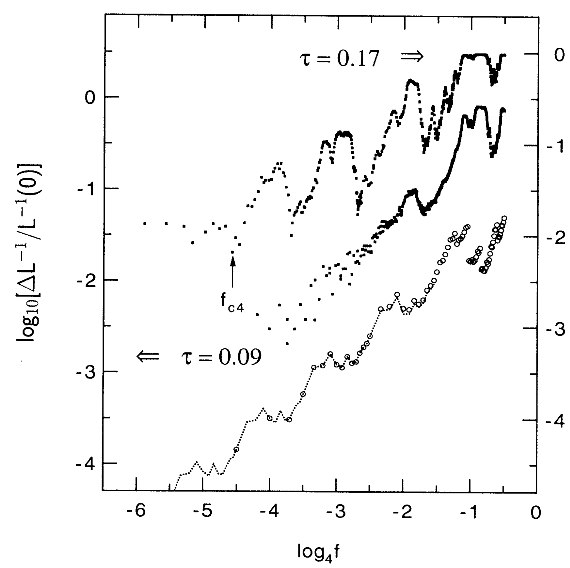

FIG. 10. log-log plot of the relative change of the inverse sheet inductance of the triangular lattice of fourth-order Sierpinski gaskets shown in Fig. 8 at two different reduced temperatures as a function of frustration. The theoretical curve (shifted downwards, for clarity, by one decade with respect to the vertical axis on the left) was computed by combining the direct method described in Sec. III (open circles) with the perturbative decimation procedure based on Eqs. (40) and (41) (dotted line). $f_{c4}=\frac{1}{512}$ is the frustration at which the crossover from the fractal ($f>f_{c4}$) to the Euclidean 2D ($f<f_{c4}$) regime occurs.

Remarkably, in spite of the indisputable evidence for fluctuation effects, the data at $\tau=0.17$ follow quite closely the asymptotic scaling prediction [Eq. (44)] of the mean-field theory. A weak upwards deviation sets in only in the last hierarchical stage ($\frac{1}{512} < f < \frac{1}{128}$) where the inductance measurements were carried out at the limit of our experimental resolution and should therefore be taken with caution. Failing a detailed theoretical description including fluctuations, we are unable to explain why their effect on the asymptotic scaling behavior turns out to be so weak.

Because of the 2D nature of our sample at length scales larger than the gasket size, at $f_{cN}=1/(2\times 4^N)=\frac{1}{512}$, the frustration defining the ground-state configuration in which each (rhombohedral) unit cell of the periodic gasket array contains just one single vortex, we expect a crossover from the fractal ($f > f_{cN}$) to the Euclidean 2D ($f < f_{cN}$) regime.[21] The tendency of the high-temperature data of Fig. 10 to flatten out below f_{c4} does indeed provide some evidence for the occurrence of a dimensional crossover. This preliminary observation has been confirmed by recent high-resolution studies of the fine structure below f_{cN} and will be discussed in more detail elsewhere.

Closer inspection of the fine structure in the data of Fig. 10 reveals a discrepancy between theory and experiment. It consists in an almost temperature-independent shift of the structures which becomes particularly manifest in the first hierarchical stage for $f > \frac{1}{4}$, but is also present at lower frustrations. We attribute this effect to the inhomogeneous frustration resulting from the change in the effective area of the different plaquettes caused by the asymmetric (with respect to the link direction) diamagnetic response of the "truncated-star-shaped" superconducting islands (see Fig. 8). Because of this particular geometrical form, the screening currents flowing in these grains create a distortion of the triangular current patterns associated with the individual loops which perturbs the self-similarity of the gaskets. Since at the temperatures of interest the magnetic penetration depth in the Pb islands is much less than their geometrical dimensions, the distortion is appreciable, thereby making f nonuniform. This interpretation is corroborated by the observation that the shift is more pronounced at higher frustrations where the loops providing the dominant contribution to the magnetoinductance, the smallest ones, turn out to be those exhibiting the largest distortion. A quantitative account of this sample-specific size effect will be published separately.

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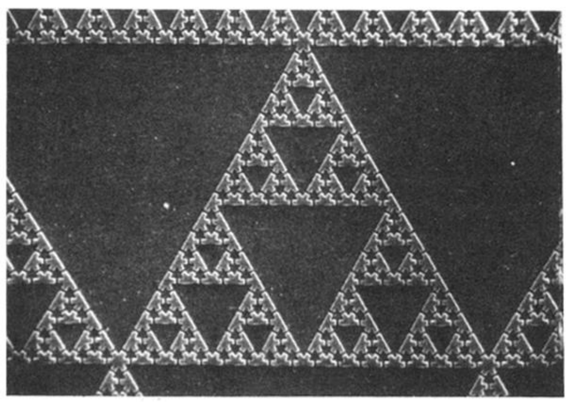

FIG. 8. Scanning electron micrograph showing a portion of a triangular array of periodically repeated fourth-order Sierpinski gaskets of proximity-effect coupled Pb/Cu/Pb Josephson junctions. The length of the elementary links of the gaskets is 8 μm. Notice that, with the exception of those centered on the common vertices of three constituent gaskets, the superconducting Pb islands have an asymmetric "truncated-star" shape.