Absence of reentrance in superconducting arrays with positional disorder

S. E. Korshunov

L. D. Landau Institute for Theoretical Physics, Kosygina 2, 117940 Moscow, Russia

Thomas Nattermann

Institut für Theoretische Physik, Universität zu Köln, D-50937, Köln, Germany
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A Josephson junction array with geometrical irregularities in the presence of a perpendicular magnetic field can be described by a two-dimensional XY model with random phase shift. If the magnitude of the field corresponds to the integer number of flux quanta per plaquette (on the average), the problem can be discussed in terms of a two-dimensional Coulomb gas interacting with a random potential, the correlations of which diverge logarithmically. In the present work we study the renormalization of the vortex interaction and random potential distribution in such a system and show that its phase diagram cannot include a low-temperature reentrant transition into a disordered phase. Our conclusions are in agreement with the results of both real and computer experiments and can be of relevance for a number of other systems.

I. INTRODUCTION

A two-dimensional superconducting array in perpendicular magnetic field can be described by the Hamiltonian of the form:

$$H = \sum_{jj'} U(\varphi_j - \varphi_{j'} - A_{jj'}), \qquad (1)$$

where φ_j is the order parameter phase on the jth superconducting island and the summation is performed over all the weak links which connect these islands. The phase shift on each link $A_{jj'}$ is determined by the vector potential of the magnetic field and in the typical experimental situation can be ascribed entirely to the external field.[1]

In the case of the Josephson junction array the interaction function $U(\theta)$ has a very simple form:

$$U(\theta) = -J \cos\theta, \qquad (2)$$

which is also typical for the XY models describing planar magnets. In the network of thin superconducting wires the form of the interaction function coincides[2] with the Villain's function $U_V(\theta)$ which can be defined by the relation

$$\exp\left[-\frac{1}{T} U_V(\theta)\right] = \sum_{m=-\infty}^{\infty} \exp\left[-\frac{J}{2T}(\theta - 2\pi m)^2\right], \quad (3)$$

and was originally introduced to simplify the analytical treatment of the models with continuous degeneracy.[3] This function is also periodic in θ and for $T \gg J$ its deviation from the cosine is exponentially small whereas in the opposite limit $T \ll J$ the form of $U_V(\theta)$ is more close to a piecewise continuation of a parabolic function:

$$U_V(\theta) \approx \begin{cases} \text{const} + \dfrac{J}{2}\, \min_{m \in Z_N}[(\theta - 2\pi m)^2] & \text{for } T \ll J \\[2ex] \text{const} - 2T \exp\left(-\dfrac{T}{2J}\right)\cos\theta & \text{for } T \gg J. \end{cases} \qquad (4)$$

In a situation when no accidental degeneracies are present such a difference in the form of the interaction between the arrays and networks is usually not expected to be of importance for the general properties of the system.

If the magnetic field is absent all variables $A_{jj'}$ are equal to zero and Hamiltonian (1) reduces to the Hamiltonian of the ordinary two-dimensional XY model which demonstrates a phase transition between a quasiordered low-temperature phase in which the correlations of φ decay algebraically (and all the vortices are bound in pairs) and the disordered high-temperature phase in which the correlations of φ decay exponentially (and vortex pairs are debound).[4-6]

The same is true for the case of finite magnetic field provided the array is regular [Fig. 1(a)] and the magnitude of the field corresponds to the integer number of superconducting flux quanta per plaquette. In that case the redefinition of the variables φ_j allows to get rid of the phase shifts $A_{jj'}$.

But if the array has geometrical irregularities [the random displacements in the positions of the superconducting islands or of the junctions, Fig. 1(b)] this is no longer possible and one has to consider $A_{jj'}$ as quenched random variables with some symmetric distribution which can be characterized by its width σ:

$$\overline{A_{jj'}} = 0; \quad \overline{A_{jj'}^2} = \sigma. \qquad (5)$$

By an overbar we denote the averages over disorder. Deviation of $A_{jj'}$ from zero signifies that the area of one of two lattice cells which are adjacent to the bond (jj') is increased by some amount whereas the area of another one is decreased by the same amount. In the first of Eqs. (5) we still assume that on the average the magnitude of the magnetic
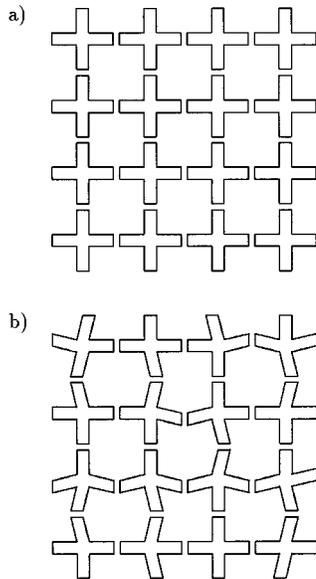
FIG. 1. A Josephson junction array consists of superconducting islands each of which is coupled to the neighboring islands by Josephson junctions. (a) Regular square array. (b) Square array with random displacements of the junctions.

field corresponds to the integer number of flux quanta per plaquette otherwise it would be impossible to make the average value of $A_{jj'}$ on each link equal to zero.

In terms of statistical mechanics the resulting model can be referred to as a two-dimensional XY model with random phase shift. Such a model (with a cosine form of the interaction) was introduced by Rubinstein *et al.*[7] for the description of a random planar magnet in which the interaction of magnetic impurities is mediated by nonmagnetic impurities (random Dzyaloshinskii-Moriya interaction) and only later it has been suggested that the same model can be applied for the description of the Josephson junction arrays with positional disorder in the presence of a perpendicular magnetic field.[8] An analogous model can be also applied for the description of the unbinding of dislocation pairs in a two-dimensional crystal with quenched impurities.[9]

The main conclusions of Ref. 7 were that for large disorder the ordered phase is always destroyed whereas for small disorder it should exist but with a decrease in temperature the reentrant transition to the disordered phase should always take place. Only the first of these conclusions has been confirmed (both in the experiments on Josephson junction arrays[10] and in numerical simulations[10–12]) whereas the reentrant transition to the disordered phase has never been observed.

Later it was shown[13] that the addition of the higher-order corrections to the consideration of Rubinstein *et al.*[7] leads to the appearence of a new divergence in each new order of the expansion and to the further suppression of the domain of stability of the ordered phase until it completely dissapears. It has been suggested in Ref. 13 that this can mean either that the ordered phase is always destroyed by the disorder (the results of the observations[10–12] in that case should be ascribed to finite-size effects) or that the correct description of the system should be based on some other approach. On the

other hand, Ozeki and Nishimori[14] have proved with the help of the formal gauge transformation that if the ordered phase of the XY model with random phase shift does exist then the phase diagram cannot include a reentrant transition into the disordered phase. Unfortunately this approach also does not allow to make any conclusions about the existence of the ordered phase.

Recently a number of arguments in favor of the existence of the ordered phase have been put forward by Nattermann *et al.*[15] which also support the absence of reentrant transition. The most transparent of them is based on comparison of the proper (elastic) energy of the vortex with the maximal energy gain it can acquire due to presence of disorder. The other approaches include the derivation of two different sets of the renormalization-group equations and the application of the Migdal-Kadanoff renormalization procedure. Some of these ideas (with the analogous conclusions) have been simultaneously introduced in the context of two-dimensional melting in the presence of a random potential by Cha and Fertig.[16]

The present work is devoted to a much more detailed presentation of one of the approaches which has been schematically sketched in Ref. 15 and is based on the systematic calculation of corrections to different quantities due to the presence of bound vortex pairs. In Sec. II we describe how the XY model with a random phase shift can be reformulated in terms of the Coulomb gas with a random potential. In Sec. III the approaches of the earlier works are presented in some detail in order to explain the problems with the fugacity expansion. In Sec. IV the general expression for the renormalized interaction of the vortices is derived. In Sec. V we introduce our main approximation which is equivalent to considering the leading correction in the concentration of bound vortex pairs. We first use this approximation to find the form of the correction to the vortex interaction. In Sec. VI the average over disorder of this correction is calculated which allows us to determine the region of stability of the ordered phase. In Sec. VII we show that the correction to the distribution of a random potential can be completely absorbed into the renormalization of the vortex interaction, the disorder strength remaining unrenormalized. The consequences of these conclusions for the shape of the phase diagram and the relation of our results with the other works are discussed in Sec. VIII.

II. THE COULOMB GAS REPRESENTATION

The statistical mechanics of two-dimensional systems with continuous degeneracy usually becomes more transparent when discussed in terms of vortices. The formal transformation from the partition function of an XY model to the partition function of a vortex gas (or a Coulomb gas, as it is usually called, since the interaction of vortices in two dimensions is logarithmic) in the case of the Villain's interaction function (3) can be performed without making any approximations.[3,17]

In that case the partition function of the XY model with random phase shift acquires the form

$$Z_{XY} = \prod_j \left(\int_{-\infty}^{\infty} d\varphi_j \right) \prod_{(jj')} \left(\sum_{m_{jj'}=-\infty}^{\infty} \right)$$

$$\times \exp\left[-\frac{J}{2T}(\varphi_j - \varphi_{j'} - A_{jj'} - 2\pi m_{jj'})^2 \right], \quad (6)$$

in which the integration with respect to the continuous variables φ_j is Gaussian and therefore can be performed exactly. That allows to reduce Eq. (6) to a partition function of a Coulomb gas, the Hamiltonian of which can be written as

$$H_{CG} = \frac{1}{2} \sum_{r,r'} s_r G_0(r-r') s_{r'} + \sum_r V(r) s_r, \quad (7)$$

where the integer variables s_r (describing the topological charges of the vortices) are defined on the sites r of a dual lattice. In terms of the integer variables $m_{jj'} \equiv -m_{j'j}$ which are defined on the lattice bonds the charges s_r are given by the relation

$$s_r = \sum_{\square r} m_{jj'}, \quad (8)$$

where $\square r$ in the summation sign indicates that the summation has to be performed over the perimeter of the plaquette which surrounds point r.

The first term in Eq. (7) describes the mutual interaction of the charges whereas the second term describes their interaction with the random potential $V(r)$. When writing Eq. (7) we have omitted a term which depends only on the random potential and does not depend on charge configuration because after averaging over disorder it drops out from all the relevant quantities.

The function $G_0(r)$ describing the interaction between the charges is proportional to the Green's function of the dual lattice and in two dimensions is logarithmically divergent. The Fourier transform of $G_0(r)$ is given by

$$G_0(q) = \frac{(2\pi)^2 J}{k^2(q)}, \quad (9)$$

where $k^2(q)$ is the Fourier transform of the Laplacian on a dual lattice (with the minus sign added). In the following we discuss only the case of a square lattice for which the dual lattice is also square and

$$k^2(q) = 4 - 2\cos q_x - 2\cos q_y. \quad (10)$$

The form of Eq. (9) implies that the energy of a single vortex is logarithmically divergent. In contrast to that the energy $g(r)$ of a neutral vortex pair is finite and for the large separation between the vortices depends logarithmically on the separation:

$$g(r) \equiv G_0(r=0) - G_0(r) \approx 2\pi J \ln|r|. \quad (11)$$

Equation (11) describes the energy of a pair formed by the charges $+1$ and -1 whereas the energy of a neutral pair formed by the charges $+s$ and $-s$ is equal to $s^2 g(r)$. In Eqs. (10) and (11) and further on we assume that the lattice constant is equal to one.

The random potential $V(r)$ is linearly related with the original random variables $A_{jj'}$:

$$V(r) = \frac{1}{2\pi} \sum_{r'} G_0(r-r') \sum_{\square r'} A_{jj'}, \quad (12)$$

and therefore if the distribution of A_{jj} is Gaussian then the distribution of $\{V(r)\}$ is also Gaussian [by $\{V(r)\}$ we denote the whole set of the variables $V(r)$]. The form of Eq. (12) allows to conclude that in terms of the Coulomb gas charges each of the random variables $A_{jj'}$ corresponds to a quenched dipole. However it will be more convenient to discuss the influence of these random dipoles in terms of the random potential $V(r)$ which they produce.

In the following we assume that the random variables $A_{jj'}$ are independent from each other and obey the Gaussian statistics. Physically the mutual independence of $A_{jj'}$ corresponds to the situation when the positions of the junctions (and not of lattice sites) experience random displacements [Fig. 1(b)]. If the parameters of the distribution are given by Eqs. (5), the Fourier transform of the correlation function describing the fluctuations of the random potential

$$D_0(q) = \sum_r \exp[-iq(r-r')] \overline{V(r)V(r')} \quad (13)$$

acquires a form

$$D_0(q) = [G_0(q)]^2 \frac{\sigma}{(2\pi)^2} k^2(q) \equiv \frac{(2\pi)^2 J^2 \sigma}{k^2(q)}, \quad (14)$$

whereas the general form of the distribution of random potential is given by the expression

$$P[\{V(r)\}] = B \exp\left\{ -\frac{1}{8\pi^2 J^2 \sigma} \sum_{(rr')} [V(r)-V(r')]^2 \right\}, \quad (15)$$

where the summation is performed over the pairs of nearest neighbors on a dual lattice and B is the normalization constant.

The form of Eqs. (14) and (15) implies that the fluctuations of the random potential $\overline{V^2(r)}$ at any point r are logarithmically divergent whereas for the difference of the potentials at two different points r_1 and r_2

$$V_{12} = V(r_1) - V(r_2), \quad (16)$$

the corresponding average

$$d(r_{12}) \equiv \overline{[V(r_1)-V(r_2)]^2} = 2[D_0(r=0) - D_0(r_{12})] \quad (17)$$

is finite but increases logarithmically with the separation $r_{12} \equiv r_1 - r_2$ between these two points:

$$d(r_{12}) \approx 4\pi J^2 \sigma \ln|r_{12}|. \quad (18)$$

In the situation when the positions of the lattice sites rather than of the junctions are subject to random displacements,[8] the correlations of the random potential also diverge logarithmically, so there is no qualitative difference between the two cases.

The expression in the right hand side of Eq. (16) can be also interpreted as a random contribution to the energy of a neutral vortex pair (with the charge $+1$ at point r_1 and the

charge -1 at point $\mathbf{r}_2$). For the sake of further reference we reproduce here the explicit expression for the probability distribution of this quantity:

$$P(\mathbf{r},V)=\int \frac{d\lambda}{2\pi} \exp\left[-\frac{d(\mathbf{r})}{2}\lambda^2+i\lambda V\right]$$

$$=[2\pi d(\mathbf{r})]^{-1/2} \exp\left[-\frac{V^2}{2d(\mathbf{r})}\right], \qquad (19)$$

which depends only on separation $\mathbf{r}$ between the vortices.

III. THE FAILURE OF THE FUGACITY EXPANSION

An expansion parameter can be formally introduced into the system by adding to Eq. (7) the term describing the core energy of the vortices:

$$\Delta H=E_c \sum_{\mathbf{r}} s_{\mathbf{r}}^2 . \qquad (20)$$

After that all the relevant quantities can be expressed in terms of the expansions in powers of the vortex fugacity

$$Y=\exp\left(-\frac{E_c}{T}\right), \qquad (21)$$

which for $E_c=0$ is equal to one. Due to logarithmic divergence of $G_0(\mathbf{R})$ only the neutral configurations of charges can give the contribution to the partition function and therefore all the expansions in Y can contain only the even order terms.

The renormalization-group analysis[6] of a phase transition in a two-dimensional Coulomb gas without disorder is based on the consideration of the lowest-order correction to the coupling constant J which according to Eq. (11) determines the long-distance asymptotics of a vortex-vortex interaction. This correction is of the second order in fugacity and in the continuous approximation can be rewritten (in our notation) as

$$\Delta J=-\pi^2 J^2 \int d^2\mathbf{R} \; R^2 W(R), \qquad (22)$$

where the weight factor $W(R)$ depends algebraically on R:

$$W(R)=\frac{2Y^2}{T} \exp\left[-\frac{g(R)}{T}\right]\propto R^{-K} \qquad (23)$$

and $K=2\pi J/T$. This correction is convergent for $K>4$ [that is for $T<(\pi/2)J$] and diverges for $K\leq 4$. The appearance of the same divergence in the next orders of the fugacity expansion leads to the shift of the transition temperature to the lower values but since no new divergencies appear the correct critical behavior can be extracted from Eq. (22) by introducing a scale-dependent coupling constant $J(R)$.

In the presence of a random potential the expression for the lowest-order correction to the coupling constant retains its form (22), whereas in the expression for the weight factor $W(R)$ the random contribution of the form (16) should be added to the energy of the vortex pair,[7]

$$W(R)=\frac{2Y^2}{T} \exp\left[-\frac{g(R)+V}{T}\right]. \qquad (24)$$

Since the distribution of V is Gaussian the averaging of Eq. (24) over disorder can be performed exactly and gives

$$\overline{W(R)}=\frac{2Y^2}{T} \exp\left[-\frac{g(R)}{T}+\frac{d(R)}{2T^2}\right]\propto R^{-K_1}, \qquad (25)$$

where

$$K_1=\frac{2\pi J}{T}-\frac{2\pi J^2\sigma}{T^2}. \qquad (26)$$

Substitution of Eqs. (25) and (26) into Eq. (22) shows that in the presence of disorder the expression for the lowest-order correction to the coupling constant is convergent only for

$$\sigma<\sigma_*; \quad T_-(J,\sigma)<T<T_+(J,\sigma), \qquad (27)$$

where

$$\sigma_*=\frac{\pi}{8}; \quad T_\pm(J,\sigma)=2J\sigma_*\left(1\pm \sqrt{1-\frac{\sigma}{\sigma_*}}\right). \qquad (28)$$

For finite fugacities the renormalization effects shift the values of the transition temperatures with respect to those given by Eqs. (28) but do not change the main conclusion of Ref. 7 — that for arbitrary small disorder the decrease in temperature should lead to a reentrant transition into the disordered phase — the same phase with debound vortex pairs as can be found at high temperatures.

In Ref. 13 the higher-order corrections to the coupling constant were considered. It turned out that in the presence of disorder the new divergence appears in each order of the fugacity expansion. Namely starting from the term of the order $2N$ a contribution appears which in terms of Eq. (22) corresponds to an averaged weight factor

$$\overline{W_{2N}(R)}\propto R^{-K_N} \qquad (29)$$

with

$$K_N=\frac{2\pi J}{T}N-\frac{2\pi J^2\sigma}{T^2}N^2. \qquad (30)$$

Note that the lowest-order exponent K_1 can be obtained from Eq. (30) by taking $N=1$.

Since the second term in Eq. (30) is negative and grows with N faster than the first term we have to conclude that for any point on the phase diagram (that is for any values of temperature and disorder) the high enough orders of the fugacity expansion contain the divergent contributions.

This can mean that the ordered phase (in which all the vortices are supposed to be bound in pairs) is always destroyed.[13] The other possibility is that the regular character of these divergencies shows only that the expansion in powers of fugacity is not the best possible expansion one can choose. In any case since the expression for the fugacity (21) has a singularity at $T=0$ the expansion in fugacity cannot necessarily be expected to be valid down to zero temperature. In the following, instead of constructing some formal expansion, we make an attempt to explicitly use the weak-

ness of interaction between different vortex pairs. Formally this corresponds to considering a lowest-order contribution in vortex pair concentration.

IV. GENERAL EXPRESSION FOR THE RENORMALIZED INTERACTION FUNCTION

The renormalized interaction function $G(\mathbf{R}_1, \mathbf{R}_2)$ can be defined as a response of the system to the introduction of two infinitely small test charges e_1 and e_2 at points $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. That means that we have to add to the Hamiltonian (7) of the Coulomb gas the proper energy of these test charges:

$$H_{\text{test}} = \tfrac{1}{2} G_0(\mathbf{R}=0)(e_1^2 + e_2^2) + G_0(\mathbf{R}_1 - \mathbf{R}_2)e_1 e_2 + V(\mathbf{R}_1)e_1$$
$$+ V(\mathbf{R}_2)e_2 \tag{31}$$

and the energy of their interaction with all the other charges

$$H_{\text{int}} = \sum_{\mathbf{r}} s_{\mathbf{r}} [G_0(\mathbf{r}-\mathbf{R}_1)e_1 + G_0(\mathbf{r}-\mathbf{R}_2)e_2]. \tag{32}$$

The renormalized interaction function $G(\mathbf{R}_1, \mathbf{R}_2)$ can then be found as the second derivative of the total free energy of the system:

$$\tilde{F} = -T \ln \tilde{Z}; \quad \tilde{Z} = \sum_{\{\mathbf{s}\}} \exp\left(-\frac{H_{\text{CG}} + H_{\text{int}} + H_{\text{test}}}{T}\right) \tag{33}$$

with respect to the values of the test charges:

$$G(\mathbf{R}_1, \mathbf{R}_2) = \frac{\partial^2 \tilde{F}}{\partial e_1 \partial e_2}\bigg|_{e_1 = e_2 = 0}. \tag{34}$$

Substitution of Eqs. (7) and (31)–(33) into Eq. (34) shows that $G(\mathbf{R}_1, \mathbf{R}_2)$ can be reduced to the form

$$G(\mathbf{R}_1, \mathbf{R}_2) = G_0(\mathbf{R}_1, \mathbf{R}_2) - \sum_{\mathbf{r}_1, \mathbf{r}_2} G_0(\mathbf{R}_1 - \mathbf{r}_1)\Sigma(\mathbf{r}_1, \mathbf{r}_2)$$
$$\times G_0(\mathbf{r}_2 - \mathbf{R}_2), \tag{35}$$

where the self-energy part $\Sigma(\mathbf{r}_1, \mathbf{r}_2)$ is proportional to the irreducible correlation function of the charges in the original Coulomb gas (without test charges)[7]

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{T}(\langle s_{\mathbf{r}_1} s_{\mathbf{r}_2}\rangle - \langle s_{\mathbf{r}_1}\rangle\langle s_{\mathbf{r}_2}\rangle), \tag{36}$$

but also can be rewritten as the second derivative of the Coulomb gas free energy with respect to the potential

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = -\frac{\partial^2 F}{\partial V(\mathbf{r}_1)\partial V(\mathbf{r}_2)}. \tag{37}$$

By angular brackets we denote the thermodynamic averages.

Analogously the thermodynamic average of a charge at a given site $\mathbf{r}$ is given by the first derivative of the free energy:

$$\langle s_{\mathbf{r}}\rangle = \frac{\partial F}{\partial V(\mathbf{r})}. \tag{38}$$

Note that Eqs. (37) and (38) were derived without making any approximation or averaging over disorder.

V. APPROXIMATION OF DILUTED PAIRS

The analysis of the phase diagram of a two-dimensional Coulomb gas is usually based on the assumption that in some domain of parameters there exists an ordered phase in which the logarithmically interacting charges are present only in the form of bound neutral pairs. And after making such assumption one has to check if the correction to the interaction of the charges due to presence of these bound pairs is important or not. In the lowest-order approximation in the pair concentration the contributions to this correction from different pairs are independent from each other. Therefore when calculating this correction the interaction betweeen the charges belonging to different pairs can be neglected.

In that case the partition function of the Coulomb gas can be rewritten in the form

$$Z = \prod_{(\mathbf{r}, \mathbf{r}')} (1 + w_{\mathbf{r}\mathbf{r}'} + w_{\mathbf{r}'\mathbf{r}}), \tag{39}$$

where

$$w_{\mathbf{r}\mathbf{r}'} = \exp\left[-\frac{g(\mathbf{r}-\mathbf{r}') + V(\mathbf{r}) - V(\mathbf{r}')}{T}\right] \tag{40}$$

and the product is assumed to be taken over all pairs of different sites on a dual lattice. The form of Eq. (39) implies that for each pair of sites $(\mathbf{r}, \mathbf{r}')$ three possibilities are considered: (i) the absence of any vortex pair (with the weight factor 1), (ii) the presence of the vortex with the topological charge $+1$ at point $\mathbf{r}$, and -1 at point $\mathbf{r}'$ (with the weight factor $w_{\mathbf{r}\mathbf{r}'}$), and (iii) the presence of the vortex with the charge $+1$ at point $\mathbf{r}'$, and -1 at point $\mathbf{r}$ (with the weight factor $w_{\mathbf{r}'\mathbf{r}}$). We exclude from the consideration the pairs formed by the charges $\pm s$ (with $s > 1$) since their energy is much larger than that of a pairs formed by the charges ± 1.

The partition function (39) takes into account in the correct form the interaction of all the vortices with the random potential and the mutual interaction of the vortices belonging to the same pair, and the only contribution which is neglected is the interaction between the vortices belonging to different pairs.

When the partition function can be decomposed into the product of independent terms the free energy can be expressed as a sum of the independent terms:

$$F = -T \sum_{(\mathbf{r}, \mathbf{r}')} \ln(1 + w_{\mathbf{r}\mathbf{r}'} + w_{\mathbf{r}'\mathbf{r}}). \tag{41}$$

Comparison with Eq. (37) shows that for any pair of different sites $\mathbf{r}_1 \neq \mathbf{r}_2$ only the one term from Eq. (41) depends both on $V(\mathbf{r}_1)$ and $V(\mathbf{r}_2)$ and therefore contributes to the expression for the self-energy part:

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{T}\frac{w_{\mathbf{r}_1\mathbf{r}_2} + w_{\mathbf{r}_2\mathbf{r}_1} + 4w_{\mathbf{r}_1\mathbf{r}_2}w_{\mathbf{r}_2\mathbf{r}_1}}{(1 + w_{\mathbf{r}_1\mathbf{r}_2} + w_{\mathbf{r}_2\mathbf{r}_1})^2}, \tag{42}$$

whereas $\Sigma(\mathbf{r}_1, \mathbf{r}_1)$ can be found from the relation

$$\sum_{\mathbf{r}_2} \Sigma(\mathbf{r}_1, \mathbf{r}_2) = 0. \tag{43}$$

Substitution of Eq. (41) into Eq. (38) shows that in the same approximation the average charge on site $\mathbf{r}$ is given by the expression

$$\langle s_\mathbf{r} \rangle = \sum_\mathbf{r} (p_{\mathbf{rr}'} - p_{\mathbf{r}'\mathbf{r}}); \quad p_{\mathbf{rr}'} = \frac{w_{\mathbf{rr}'}}{1 + w_{\mathbf{rr}'} + w_{\mathbf{r}'\mathbf{r}}}, \quad (44)$$

which has a very transparent meaning since $p_{\mathbf{rr}'}$ and $p_{\mathbf{r}'\mathbf{r}}$ can be interpreted as the normalized probabilities to find a neutral vortex pair (with a particular orientation) at a given pair of sites $\mathbf{r}$ and $\mathbf{r}'$. So it looks not unexpected that the average charge at point $\mathbf{r}$ is given by the difference between the probability to have some pair with a positive charge at point $\mathbf{r}$ and the probability to have a pair with the negative charge at the same point. In contrast to that Eq. (42), which in terms of the normalized probabilities can be rewritten as

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{T}[p_{\mathbf{r}_1\mathbf{r}_2} + p_{\mathbf{r}_2\mathbf{r}_1} - (p_{\mathbf{r}_1\mathbf{r}_2} - p_{\mathbf{r}_2\mathbf{r}_1})^2], \quad (45)$$

reduces to the intuitively expected form

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{T}(p_{\mathbf{r}_1\mathbf{r}_2} + p_{\mathbf{r}_2\mathbf{r}_1}) \quad (46)$$

only for $p_{\mathbf{r}_1\mathbf{r}_2} = p_{\mathbf{r}_2\mathbf{r}_1}$, that is, in the absence of disorder.

Since the distribution of the random potential is translationally invariant the self-energy function $\Sigma(\mathbf{r}_1, \mathbf{r}_2)$ after averaging over disorder can depend only on $\mathbf{r}_1 - \mathbf{r}_2$. That allows to rewrite Eq. (35) in the Fourier representation as follows:

$$G(\mathbf{q}) = G_0(\mathbf{q}) - G_0^2(\mathbf{q})\overline{\Sigma}(\mathbf{q}), \quad (47)$$

where the Fourier transform of the self-energy part $\overline{\Sigma}(\mathbf{q})$ in the aproximation of diluted pairs has the form

$$\overline{\Sigma}_2(\mathbf{q}) = \sum_\mathbf{R} (1 - \cos\mathbf{qR})\overline{W(\mathbf{R})}. \quad (48)$$

The disorder-averaged weight factor $\overline{W(\mathbf{R})}$ in Eq. (48) is given by the integral:

$$\overline{W(\mathbf{R})} = \int_{-\infty}^{\infty} dV P(\mathbf{R}, V) W(\mathbf{R}, V), \quad (49)$$

in which the distribution function $P(\mathbf{R}, V)$ is defined by Eq. (19). The form of the potential-dependent weight factor $W(\mathbf{R}, V) \equiv -\Sigma(\mathbf{r} + \mathbf{R}, \mathbf{r})$ is determined by Eq. (42) and can be rewritten as

$$W(\mathbf{R}, V) = \frac{\partial}{\partial V} f(\mathbf{R}, V), \quad (50)$$

where

$$f(\mathbf{R}, V) = \frac{w(\mathbf{R}, -V) - w(\mathbf{R}, V)}{1 + w(\mathbf{R}, V) + w(\mathbf{R}, -V)}, \quad (51)$$

whereas

$$w(\mathbf{R}, V) = \exp\left[-\frac{g(\mathbf{R}) + V}{T} \right] \quad (52)$$

is just expression (40) rewritten in the simplified notation.

When only the long-distance behavior of the vortex interaction is important the correction to the interaction can be expressed as a correction to a coupling constant J:

$$\Delta J = -\pi^2 J^2 \sum_\mathbf{R} R^2 \overline{W(\mathbf{R})}, \quad (53)$$

where the same expression (49) for the disorder-averaged weight factor should be used.

Thus we have found the form of a leading correction to the coupling constant which is proportional to the concentration of the vortex pairs. In the limit of a large core energy this correction dominates over all the higher-order corrections and the position of the phase transition line is given by the line on which this correction diverges. When the core energy is finite the important sequence of higher-order corrections can be taken into account by considering the coupling constant as being scale dependent. The renormalization of the coupling constant then leads to the shift of the phase transition line to the lower temperatures.

VI. AVERAGING OF THE WEIGHT FACTOR OVER DISORDER

In order to find the R dependence of the disorder-averaged weight factor $\overline{W(R)}$ one has to calculate the integral in Eq. (49). The results of the previous works can be reproduced by choosing a particular approximation to perform this calculation. If the last two terms in the denominator of Eq. (51) are completely neglected (on the assumption that they are always much smaller than one) the integration in Eq. (49) becomes Gaussian and can be performed exactly reproducing Eq. (25) which corresponds to the results of Rubinstein et al.[7]

If instead of neglecting the corrections to 1 in the denominator of Eq. (51) one makes an attempt to take them into account by expanding Eq. (51) in powers of $[w(\mathbf{R}, V) + w(\mathbf{R}, -V)]$:

$$f(\mathbf{R}, V) = [w(\mathbf{R}, -V) - w(\mathbf{R}, V)] \sum_{N=1}^{\infty} (-1)^{N-1}$$
$$\times [w(\mathbf{R}, V) + w(\mathbf{R}, -V)]^{N-1}, \quad (54)$$

it becomes possible to perform the integration in Eq. (49) separately for each term of this series, but substitution of the results in Eq. (53) then reproduces exactly the same set of divergencies as was found in Ref. 13. This gives a hope that our derivation of the lowest-order correction in vortex pair concentration is equivalent to the effective resummation of the divergencies which appear in the fugacity expansion.

In the limit of zero temperature the function $f(\mathbf{R}, V)$ defined by Eqs. (51) and (52) acquires a steplike form

$$f(\mathbf{R}, V) = \theta[V - g(\mathbf{R})] - \theta[-V - g(\mathbf{R})], \quad (55)$$

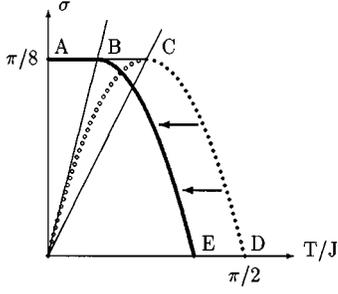and the integration in Eq. (49) can be performed exactly producing the expression

FIG. 2. Renormalization effects shift the curved part of the phase transition line to the lower temperatures. According to the results of Ozeki and Nishimori (Ref. 14) for $T/J < \sigma$ the phase transition line should remain parallel to the temperature axis.

$$\overline{W(\mathbf{R})} = P[\mathbf{R}, g(\mathbf{R})] + P[\mathbf{R}, -g(\mathbf{R})]$$

$$= \sqrt{\frac{2}{\pi d(\mathbf{R})}} \exp\left[-\frac{g^2(\mathbf{R})}{2d(\mathbf{R})}\right], \quad (56)$$

which for $R \to \infty$ is also characterized by an algebraic behavior

$$\overline{W(R)} \approx \frac{1}{\pi J \sqrt{2\sigma \ln R}} R^{-K_0}; \quad K_0 = \frac{\pi}{2\sigma}. \quad (57)$$

For $T > 0$ only the asymptotic behavior of $\overline{W(R)}$ for $R \to \infty$ [when both $g(R)$ and $d(R)$ are logarithmically large] can be found analytically. It turns out that it is always algebraic but the exponent and the prefactor are given by different expressions depending on whether the temperature T is larger or smaller than $T_*(J, \sigma) \equiv 2J\sigma$. For $T > T_*(J, \sigma)$ the result (25) of Ref. 7 is reproduced, whereas for $T < T_*(J, \sigma)$ the result of the averaging differs from the zero-temperature result (57) only by the appearance of the temperature-dependent prefactor

$$\overline{W(R)} \approx \frac{\pi T / T_*}{\sin(\pi T / T_*)} \frac{1}{\pi J \sqrt{2\sigma \ln R}} R^{-K_0}, \quad (58)$$

whereas the value of the exponent K_0 remains independent of temperature.

For $T = T_*(J, \sigma)$ the values of the exponents K_0 and K_1 coincide with each other and therefore the crossover between two regimes is rather smooth. For $T \to T_* - 0$ and fixed $R \gg 1$ Eq. (58) gives the correct value of the preexponential factor until it remains much smaller than the preexponential factor $2/T$ of the other regime. That means that the width ΔT of a temperature region in which the crossover between the two expressions for the preexponential factor takes place decays with increase in R as

$$\frac{\Delta T}{T_*} \propto \sqrt{\frac{\sigma}{\ln R}}. \quad (59)$$

Substitution of the results of this section into Eq. (53) shows that for $T > T_*(J, \sigma)$ the lowest-order correction to the coupling constant becomes divergent on the same line $T = T_+(J, \sigma)$ (the line CD in Fig. 2) as was found by Rubin-

stein et al.,[7] whereas at low temperatures $[T < T_*(J, \sigma)]$ the domain of convergence is increased and is restricted by the line $\sigma = \sigma_* = \pi/8$ (the line AC in Fig. 2) which is parallel to the temperature axis. The more attentive consideration shows that $\overline{W(R)}$ always decreases with a decrease in temperature and therefore there are no reasons to expect a reentrant transition.

The calculation presented in Ref. 15 in terms of the polarizability of vortex pairs is equivalent to substitution of Eq. (45) by a more simple but less justified Eq. (46). However this substitution changes only the prefactor in Eq. (58) whereas the exponents K_0 and K_1 (and therefore the domain where the correction to the coupling constant is convergent) turn out to be the same as given by a more rigorous calculation presented here.

VII. RENORMALIZATION OF THE RANDOM POTENTIAL DISTRIBUTION

In the Coulomb gas representation the considered problem is characterized by the presence of two logarithmically divergent quantities — the vortex interaction $G_0(\mathbf{R})$ and a random potential correlation function $D_0(\mathbf{R})$. The appearance of the bound vortex pairs (due to thermal fluctuations and/or disorder) can lead to the renormalization of both of them and now we have to consider how the distribution of the random potential is renormalized.

Analogously to the definition of the renormalized interaction function introduced in Sec. IV the renormalized potential $V_R(\mathbf{r})$ can be defined as a response of the system to the introduction of one infinitely small test charge e at point $\mathbf{r}$:

$$V_R(\mathbf{r}) = \frac{\partial \tilde{F}}{\partial e}\bigg|_{e=0} = V(\mathbf{r}) + v(\mathbf{r});$$

$$v(\mathbf{r}) \equiv \sum_{\mathbf{r}'} G(\mathbf{r} - \mathbf{r}') \langle s_{\mathbf{r}'} \rangle. \quad (60)$$

Since in the absence of disorder [for $V(\mathbf{r}) = $ const] the average charge on each site is equal to zero, the random potential cannot appear spontaneously and can only be renormalized if it is already present.

The result of the substitution of Eq. (60) into the expression for the Fourier transform of the correlation function

$$D(\mathbf{q}) = \sum_{\mathbf{r}_1} \exp[-i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)] \overline{V_R(\mathbf{r}_1) V_R(\mathbf{r}_2)}, \quad (61)$$

can be naturally decomposed into three terms

$$D(\mathbf{q}) = D_0(\mathbf{q}) + D_1(\mathbf{q}) + D_2(\mathbf{q}), \quad (62)$$

the first of which [given by Eq. (14)] is trivial and describes the correlations of the bare random potential.

The second term describing the correlations between the bare potential $V(\mathbf{r})$ and the induced potential $v(\mathbf{r})$ can be rewritten in the form

$$D_1(\mathbf{q}) = 2G_0(\mathbf{q}) \sum_{\mathbf{r}_1} \cos \mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2) \overline{V(\mathbf{r}_1) \langle s_{\mathbf{r}_2} \rangle}, \quad (63)$$

where $\overline{V(\mathbf{r}_1) \langle s_{\mathbf{r}_2} \rangle}$ is given by the integral

$$\overline{V(\mathbf{r}_1) \langle s_{\mathbf{r}_2} \rangle} = \prod_{\mathbf{R}} \left[\int_{-\infty}^{\infty} dV(\mathbf{R}) \right] P[\{V\}] V(\mathbf{r}_1) \frac{\partial}{\partial V(\mathbf{r}_2)} F \quad (64)$$

and $P[\{V\}]$ is the distribution function of the bare random potential defined by Eq. (15).

Differentiation of Eq. (15) with respect to $V(\mathbf{r}_3)$ produces a relation

$$\frac{\partial}{\partial V(\mathbf{r}_3)} P[\{V\}] = -P[\{V\}] \sum_{\mathbf{r}_4} D_0^{-1}(\mathbf{r}_3 - \mathbf{r}_4) V(\mathbf{r}_4), \quad (65)$$

which after multiplying it by $D_0(\mathbf{r}_1 - \mathbf{r}_3)$ and summing over $\mathbf{r}_3$ can be rewritten as

$$P[\{V\}] V(\mathbf{r}_1) = -\sum_{\mathbf{r}_3} D_0(\mathbf{r}_1 - \mathbf{r}_3) \frac{\partial P[\{V\}]}{\partial V(\mathbf{r}_3)}. \quad (66)$$

Substitution of Eq. (66) into Eq. (64) and application of the integration by parts with respect to the variable $V(\mathbf{r}_3)$ allows to rewrite Eq. (64) as

$$\overline{V(\mathbf{r}_1) \langle s_{\mathbf{r}_2} \rangle} = -\sum_{\mathbf{r}_3} D_0(\mathbf{r}_1 - \mathbf{r}_3) \overline{\Sigma}(\mathbf{r}_3, \mathbf{r}_2), \quad (67)$$

where $\overline{\Sigma}(\mathbf{r}_3, \mathbf{r}_2)$ stands for exactly the same expression (37) for the self-energy function as was derived in Sec. IV. Since $\overline{\Sigma}(\mathbf{r}_3, \mathbf{r}_2)$ can depend only on $\mathbf{r}_3 - \mathbf{r}_2$ the result of the substitution of Eq. (67) into Eq. (63) can be rewritten as

$$D_1(\mathbf{q}) = -2G_0(\mathbf{q}) D_0(\mathbf{q}) \overline{\Sigma}(\mathbf{q}). \quad (68)$$

Note that Eqs. (67) and (68) were derived without making any approximations.

The last term in Eq. (62) describing the correlations of the induced potential $v(\mathbf{r})$ in different points can be rewritten as

$$D_2(\mathbf{q}) = G_0^2(\mathbf{q}) \sum_{\mathbf{r}_1} \exp[-i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)] \overline{\langle s_{\mathbf{r}_1} \rangle \langle s_{\mathbf{r}_2} \rangle} \quad (69)$$

and to calculate the reducible part of the charge-charge correlation function which enters Eq. (69), we have to resort to the same approximation of diluted pairs.

With the help of Eq. (44) the expression for $\overline{\langle s_{\mathbf{r}_1} \rangle \langle s_{\mathbf{r}_2} \rangle}$ can be rewritten as

$$\overline{\langle s_{\mathbf{r}_1} \rangle \langle s_{\mathbf{r}_2} \rangle} = \sum_{\mathbf{r}_3, \mathbf{r}_4} \int_{-\infty}^{\infty} dV_{13} \int_{-\infty}^{\infty} dV_{24} \, P(V_{13}, V_{24})$$

$$\times f(\mathbf{r}_1 - \mathbf{r}_3, V_{13}) f(\mathbf{r}_2 - \mathbf{r}_4, V_{24}), \quad (70)$$

where $f(\mathbf{r}, V)$ is given by Eq. (51) and $P(V_{13}, V_{24})$ is the distribution function describing the joint distribution of two random quantities:

$$V_{13} = V(\mathbf{r}_1) - V(\mathbf{r}_3), \quad (71)$$

and

$$V_{24} = V(\mathbf{r}_2) - V(\mathbf{r}_4), \quad (72)$$

which can be interpreted as the random contributions to the energies of two different vortex pairs. This distribution function can be obtained from the general distribution function (15) by imposing the constraints (71) and (72) with the help of integration over two Lagrange multipliers λ_{13} and λ_{24} and integrating out all the variables $V(\mathbf{r})$. This gives

$$P(V_{13}, V_{24}) = \int_{-\infty}^{\infty} \frac{d\lambda_{13}}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda_{24}}{2\pi} \exp\left(-\frac{d_{13}}{2} \lambda_{13}^2 - d_{int} \lambda_{13} \lambda_{24} \right.$$

$$\left. -\frac{d_{24}}{2} \lambda_{24}^2 + i\lambda_{13} V_{13} + i\lambda_{24} V_{24} \right), \quad (73)$$

where

$$d_{13} = 2D_0(\mathbf{r} = 0) - 2D_0(\mathbf{r}_1 - \mathbf{r}_3), \quad (74)$$

$$d_{24} = 2D_0(\mathbf{r} = 0) - 2D_0(\mathbf{r}_2 - \mathbf{r}_4), \quad (75)$$

$$d_{int} = D_0(\mathbf{r}_1 - \mathbf{r}_2) - D_0(\mathbf{r}_2 - \mathbf{r}_3) - D_0(\mathbf{r}_1 - \mathbf{r}_4) + D_0(\mathbf{r}_3 - \mathbf{r}_4). \quad (76)$$

Since for the large separation R between the two pairs of sites $(\mathbf{r}_1, \mathbf{r}_3)$ and $(\mathbf{r}_2, \mathbf{r}_4)$ (that is for $|\mathbf{r}_1 - \mathbf{r}_3|, |\mathbf{r}_2 - \mathbf{r}_4| \ll |\mathbf{r}_1 - \mathbf{r}_2| \sim R$) the quantity d_{int} defined by Eq. (76) decays as $1/R^2$, Eq. (73), can be expanded in powers of the term $-d_{int}\lambda_{13}\lambda_{24}$ which couples λ_{13} and λ_{24}. In each term of the resulting series the integrations with respect to λ_{13} and λ_{24} can be then performed independently from each other which gives

$$P(V_{13}, V_{24}) = P(V_{13}) P(V_{24}) \left[1 + \frac{d_{int}}{d_{13} d_{24}} \right.$$

$$+ \frac{1}{2} \left(\frac{d_{int}}{d_{13} d_{24}} V_{13} V_{24} \right)^2 (d_{13} - V_{13}^2)(d_{24} - V_{24}^2)$$

$$\left. + \cdots \right], \quad (77)$$

where $P(V_{ab}) \equiv P(\mathbf{r}_a - \mathbf{r}_b, V_{ab})$ are the distribution functions of the form (19) which were introduced when discussing the properties of a single pair.

When Eq. (77) is substituted in Eq. (70) the contribution from the first term of Eq. (77) proves to be zero [just because $f(\mathbf{R}, V)$ is odd in V], whereas the Fourier transform of the contribution from the second term has a form

$$\overline{\langle s \rangle \langle s \rangle}(\mathbf{q}) \equiv \sum_{\mathbf{r}_1} \exp[-i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)] \overline{\langle s_{\mathbf{r}_1} \rangle \langle s_{\mathbf{r}_2} \rangle}$$

$$= D_0(\mathbf{q}) [\overline{\Sigma}_2(\mathbf{q})]^2, \quad (78)$$

where $\overline{\Sigma}_2(\mathbf{q})$ is given by expression (48) which describes the self-energy function in the approximation of diluted pairs. Since $d_{int} \propto 1/R^2$ the higher-order terms in Eq. (77) decay with R more quickly than the second term and therefore cannot be expected to be of importance for the renormalization of the divergent part of the random potential correlation function.

Although the distribution of a random potential is renormalized all the corrections which we have found insofar can be absorbed into the renormalization of the vortex interaction whereas the parameter σ_R which describes the effective strength of disorder and in general situation by the analogy with Eq. (14) can be defined with the help of the relation:

$$\sigma_R = \lim_{\mathbf{q} \to 0} \frac{D(\mathbf{q})}{G^2(\mathbf{q})[k^2(\mathbf{q})/(2\pi)^2]} \qquad (79)$$

remains unrenormalized. With the help of Eqs. (14), (62), (68), and (69), Eq. (79) can be transformed to the form

$$\sigma_R = \sigma \left\{ 1 + \lim_{\mathbf{q} \to 0} \frac{G_0^4(\mathbf{q})}{G^2(\mathbf{q})} \left[-\overline{\Sigma}^2(\mathbf{q}) + \frac{1}{D_0(\mathbf{q})} \overline{\langle s \rangle \langle s \rangle}(\mathbf{q}) \right] \right\} \qquad (80)$$

which demonstrates that the first order correction to σ is always absent [the origin of this property can be traced back to the relation (68) which does not rely on any approximation]. Moreover, in the lowest order in vortex pair concentration $\overline{\langle s \rangle \langle s \rangle}(\mathbf{q})$ is given by Eq. (78) whereas $\overline{\Sigma}(\mathbf{q}) \approx \overline{\Sigma}_2(\mathbf{q})$. Therefore the only two terms in Eq. (80) which are of the second order in vortex pair concentration cancel out and any nonvanishing contribution can appear only in the third order.

The absence of the renormalization of σ in the lowest order in vortex fugacity has been demonstrated by Rubinstein *et al.*[7] Our derivation shows that this property still persists on a more general level of consideration. And when some quantity remains unrenormalized in two orders of some expansion it is very probable that it is not renormalized at all.

VIII. PHASE DIAGRAM

Thus we have shown that the disorder strength σ is not renormalized at least in the two lowest orders of the expansion in vortex pair concentration, whereas the correction to the coupling constant J remains convergent for

$$\sigma < \sigma_*; \quad T < T_+(J,\sigma) \qquad (81)$$

that is below the line $ABCD$ in Fig. 2. That describes the domain of stability of the ordered phase in the limit of infinite core energy ($E_c \to \infty$). For finite or zero core energy the renormalization effects will be important and the domain of the ordered phase will shrink. But since only J is renormalized and σ remains fixed the position of the low-temperature part of the phase-transition line ($\sigma = \sigma_*$) will not be changed whereas the other part of the phase-transition line (CD) will be shifted to lower temperatures (BE).

The main conclusion of this work is that in the situation when the XY model with a random phase shift can be described in terms of a Coulomb gas with a random potential the reentrant transition to the disordered phase cannot take place, since the correctly calculated average of the correction to vortex-vortex interaction always decreases with the decrease in temperature and therefore cannot become more important. This conclusion is in agreement with the results of experiments on Josephson junction arrays[10] and of numerical simulations[10–12] as well as with Ozeki-Nishimori theorem.[14]

The more subtle point is the shape of the phase-transition line. Ozeki and Nishimori have proved[14] that at low temperatures it should contain a straight segment parallel to the temperature axis [$\sigma_c(T) = $ const]. Note however that the derivation presented in Ref. 14 works only if the interaction in the XY-type Hamiltonian [Eq. (1)] and the distribution of the random potential have the same functional dependence, otherwise the gauge transformation used in Ref. 14 cannot be applied. The analysis of our work assumes that the interaction function is of the Villain's form (3), whereas the distribution of the random phase shift is Gaussian. This particular case belongs to the class of systems for which the analysis of Ref. 14 is applicable. That is why in Fig. 2 we have shown that the singular point C is shifted by the renormalization effects to the position B where the line $\sigma = \sigma_*$ crosses the Nishimori line $T = J\sigma$.

Unfortunately both real[10] and computer[10–12] experiments were not accurate enough to allow any conclusions about the existence of such a remarkable feature as a partition of the phase-transition line into two segments (one of which is perfectly straight). Moreover in all these works the situation with random displacements of the lattice sites was considered which corresponds to the existence of short-range correlations between the random variables $A_{jj'}$ and does not allow for the application of the gauge transformation considered in Ref. 14.

Thus only the further more attentive simulations can give the answer if the structure of the phase-transition line is sensitive to such details as the exact form of the interaction in the XY-type Hamiltonian (1) or the presence of short-range correlations between quenched variables. Our scenario can be compatible with the absence of a straight segment on the phase-transition line if the singular point C is shifted to the left until it coincides with point A. But even in that case the critical value of disorder at zero temperature should remain the same.

When the phase-transition line consists of two segments, one of which is straight and the other is curved, the point which separates them should be a singular point. The critical behavior on these two segments cannot be the same. In Ref. 15 the two different sets of renormalization-group equations were discussed which should be valid for $T < T_*$ and $T > T_*$ if the next orders of the expansion in vortex pair concentration do not produce any new divergencies. But the most crucial word in the previous statement is *if*. To achieve a better understanding of the problem it will be necessary to check if the systematic application of the expansion in vortex pairs concentration really is equivalent to a complete summation of all the divergencies which appear in the fugacity expansion[13] or maybe some traces of them still survive in the higher orders. The other important direction of further investigation may consist in looking for a way to reproduce the same results in the framework of replica representation. Note that the fugacity expansion can be constructed both in replica representation and by averaging the perturbation expansion in powers of a random potential.[7,13]

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